Mineral Physics I
Chapter 2. Elasticity
Section 9. Acoustic impedance

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Wave reflection

- Reflection of seismic waves
  - if the structure is discontinuous
    - ✓ Mohorovičić discontinuity, 660-km discontinuity, core-mantle boundary
- In what conditions the wave is reflected?
    - ✓ Or what is the critical parameter to judge reflectivity?
  - Large difference in acoustic impedance between two media
    - ✓ $Z_s = \rho v$ (2.9.1)
      - ρ: density, v: velocity

Energy of 1D wave -1

- The real part of the wave equation:
  - $u = u_0 \cos(kx - \omega t)$ (2.0.16')
  - $\frac{\partial u}{\partial x} = -k u_0 \sin(kx - \omega t)$ (2.9.2)
  - $\frac{\partial u}{\partial t} = \omega u_0 \sin(kx - \omega t)$ (2.9.3)
- The potential (strain) energy, $U_{\mathrm{P}} = \frac{1}{2} E \varepsilon^2$,
  - $U_p = \frac{E}{2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{E k^2 u_0^2}{2} \sin^2(kx - \omega t) = \frac{E k^2 u_0^2}{2} \frac{1-\cos 2(kx-\omega t)}{2}$ (2.9.4)
- The kinetic energy, $U_{\mathrm{K}} = \frac{1}{2} \rho \dot{u}^2$,
  - $U_k = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 = \frac{\rho \omega^2 u_0^2}{2} \sin^2(kx - \omega t) = \frac{\rho \omega^2 u_0^2}{2} \frac{1-\cos 2(kx-\omega t)}{2}$ (2.9.5)

Energy of 1D wave -2

- The total energy $U_T = U_\mathrm{P} + U_\mathrm{K}$
  - $U_T = \frac{Ek^2u_0^2}{2}\frac{1-\cos 2(kx-\omega t)}{2} + \frac{\rho\omega^2u_0^2}{2}\frac{1-\cos 2(kx-\omega t)}{2}$

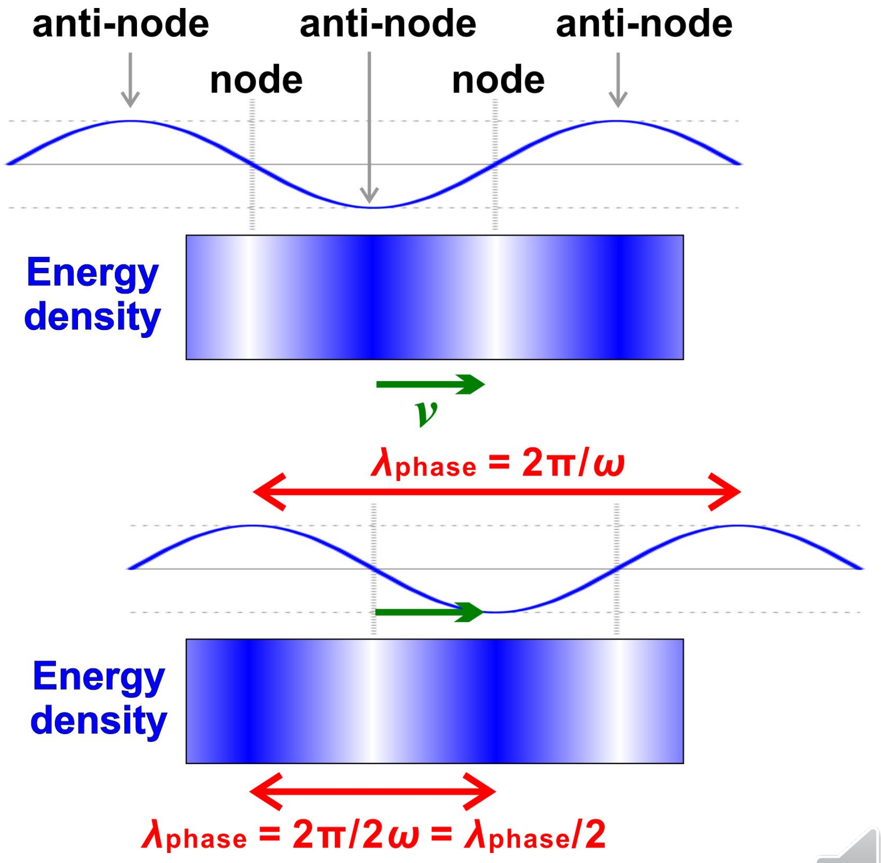
$$= \frac{1}{2}[Ek^2 + \rho\omega^2]u_0^2\frac{1-\cos 2(kx-\omega t)}{2}$$

$$= \frac{1}{2}\left[\frac{E}{\rho}\left(\frac{k}{\omega}\right)^2 + 1\right]\rho\omega^2u_0^2\frac{1-\cos 2(kx-\omega t)}{2}$$

$$= \frac{1}{2}\left[v^2\left(\frac{1}{v}\right)^2 + 1\right]\rho\omega^2u_0^2\frac{1-\cos 2(kx-\omega t)}{2}$$

$$= \rho\omega^2u_0^2\frac{1-\cos 2(kx-\omega t)}{2} \quad (2.9.6)$$

- ✓ because $v = f\lambda = \omega/k$, $v = \sqrt{E/\rho}$



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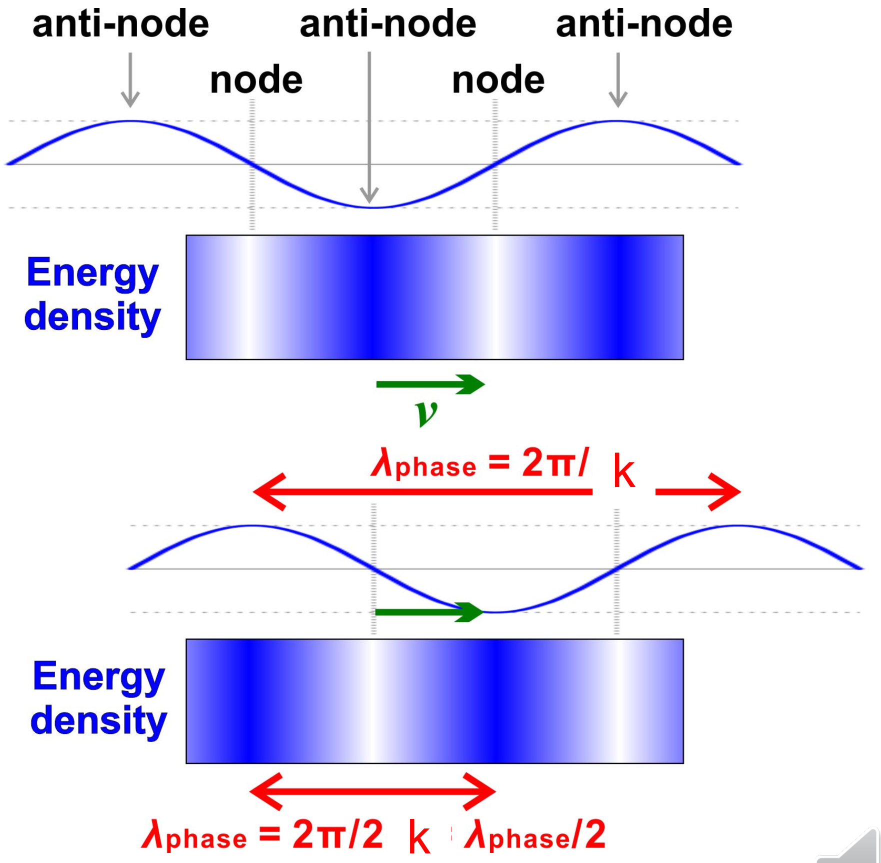
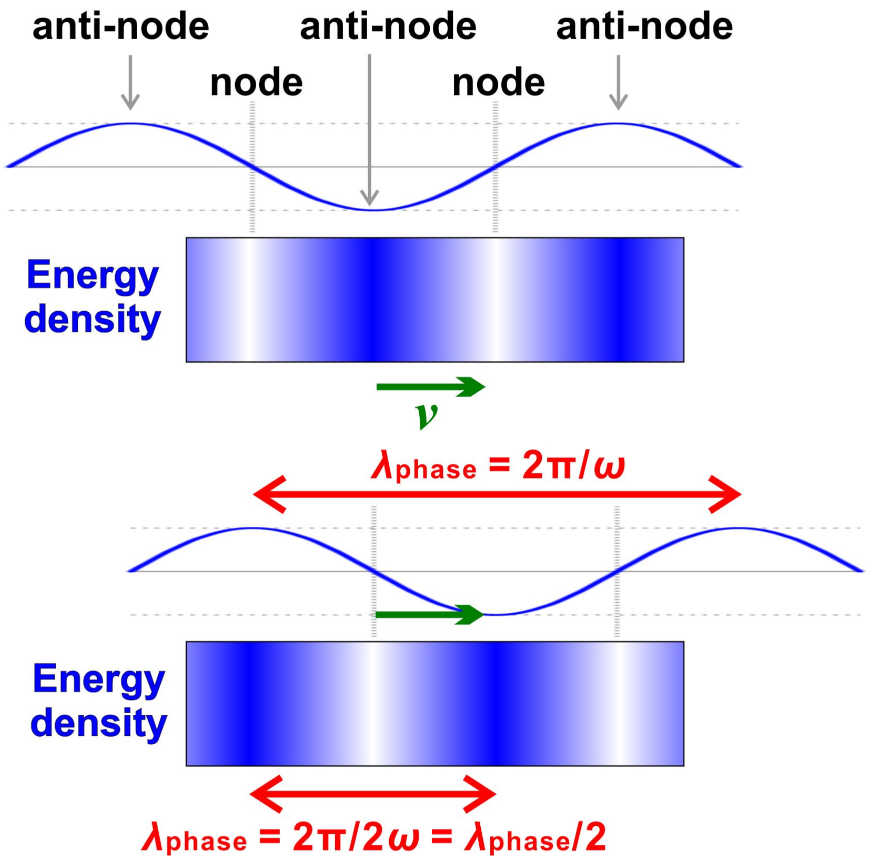


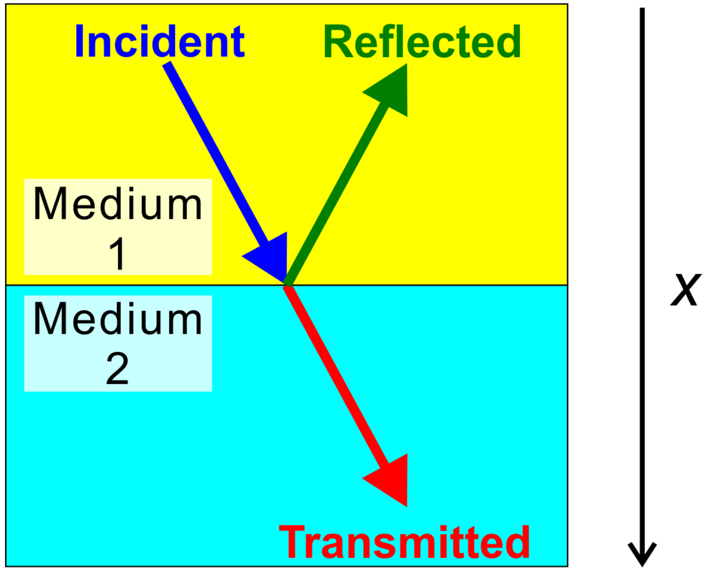
Energy of 1D wave -3

- The average energy of a wave proportional to ρ, ω^2 and u_0^2
- $U_T \propto \frac{1-\cos 2(kx-\omega t)}{2}$, i.e., not kept constant within a given volume element.
  - ✓ Energy maxima at $\cos(kx-\omega t) = \pm 1$ or $\cos(2kx-2\omega t) = -1$ (**anti-node**)
  - ✓ Energy minima (zero) at $\cos(kx-\omega t) = 0$ or $\cos(2kx-2\omega t) = +1$ (**node**)
- Energy is transferred by movement of nodes
  - ✓ Transferred energy per unit time is proportional to the wave velocity
    - $\rho\omega^2 u_0^2 \cdot v$ (2.9.7)



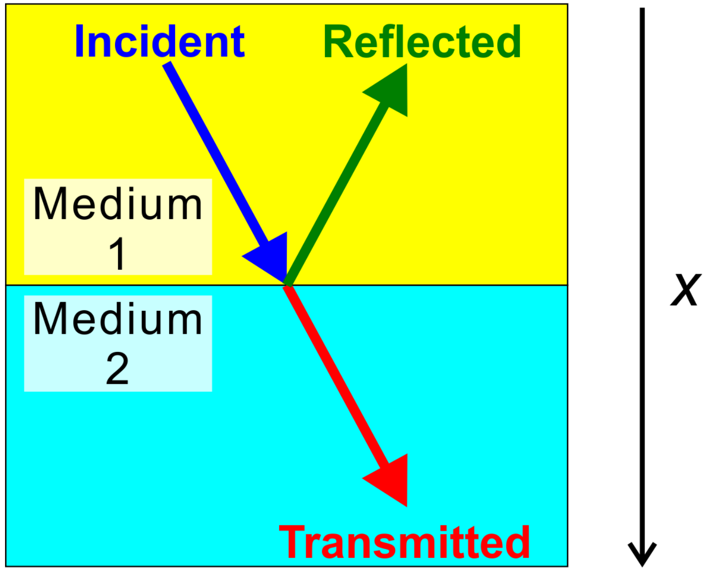
Reflection and transmission -1

- Setting
  - Medium 1: upper side, Medium 2: lower side
  - The downward direction: $+x$
  - The boundary of Media 1 and 2: $x = 0$
  - Incident wave from Medium 1 side, transmitted to Medium 2, reflected at the boundary in the Medium 1



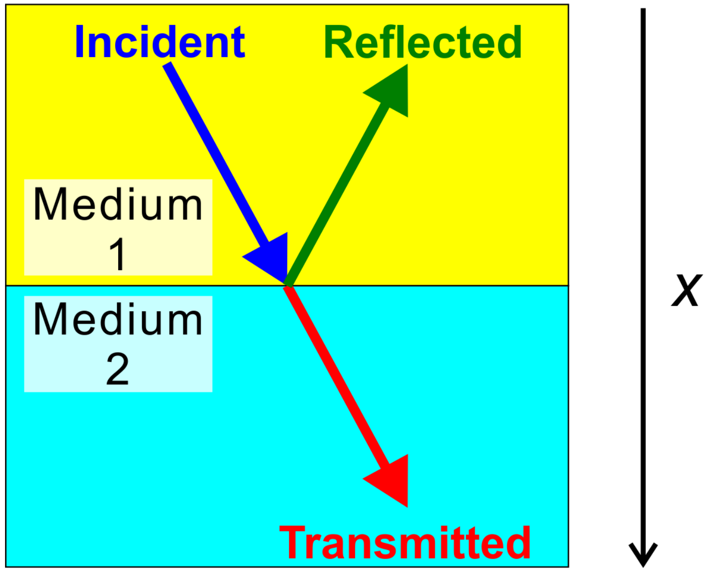
Reflection and transmission -2

- Notation:
  - $u_i(x,t)$: the displacement by the incident wave at point x and time t
  - $u_r(x,t)$: the displacement by the reflected wave at point x and time t
  - $u_t(x,t)$: the displacement by the transmitted wave at point x and time t
  - u_{i0}: the amplitude of the incident wave
  - u_{r0}: the amplitude of the reflected wave
  - u_{t0}: the amplitude of the transmitted wave
  - v_1: the velocity in Medium 1
  - v_2: the velocity in Medium 2



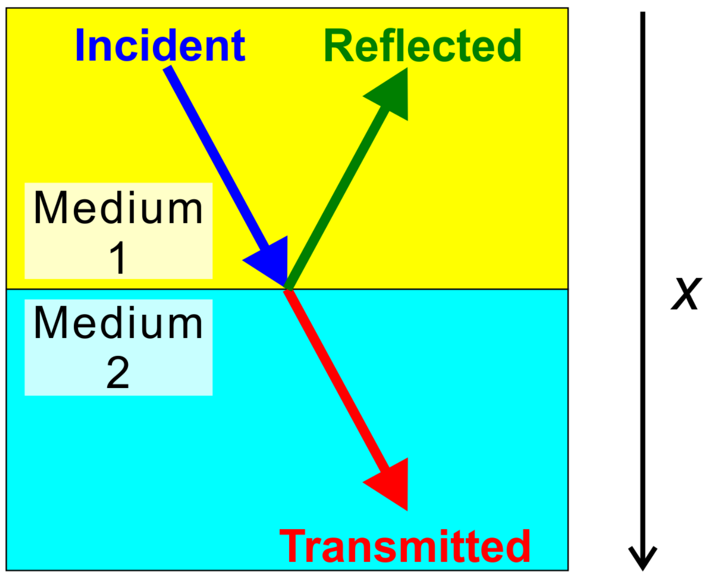
Reflection and transmission -3

- The displacements of both media at the boundary must be equal:
  - $u_i(0,t) + u_r(0,t) = u_t(0,t)$
  - $u_{i0} + u_{r0} = u_{t0}$ (2.9.9)
- The conservation of energy:
  - $\rho_1 \omega^2 u_{i0}^2 v_1 = \rho_1 \omega^2 u_{r0}^2 v_1 + \rho_2 \omega^2 u_{t0}^2 v_2$ (2.9.10)
- By substituting (2.9.9) into (2.9.10), we have
  - $\rho_1 u_{i0}^2 v_1 = \rho_1 u_{r0}^2 v_1 + \rho_2 (u_{i0} + u_{r0})^2 v_2$ (2.9.11)



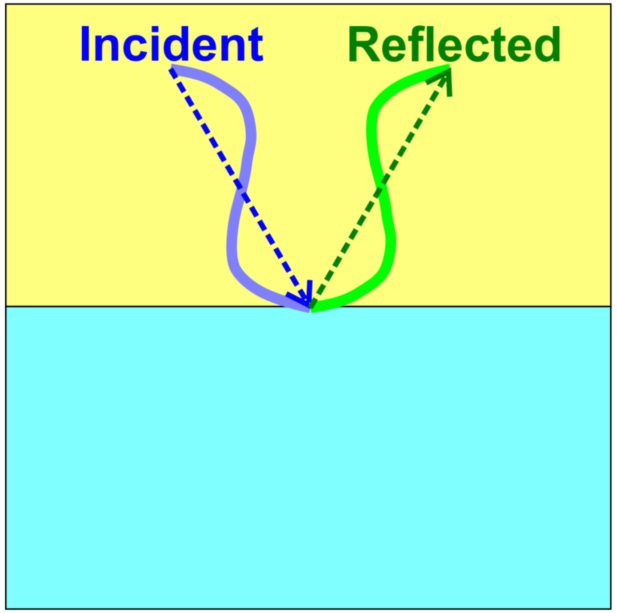
Reflection and transmission -4

- By substituting (2.9.10) into (2.9.11), we have had
  - $\rho_1 u_{i0}^2 v_1 = \rho_1 u_{r0}^2 v_1 + \rho_2 (u_{i0} + u_{r0})^2 v_2$
  - $\rho_1 v_1 u_{i0}^2 - \rho_1 v_1 u_{r0}^2 - \rho_2 v_2 u_{i0}^2 - 2\rho_2 v_2 u_{i0} u_{r0} - \rho_2 v_2 u_{r0}^2 = 0$
  - $\rho_1 v_1 u_{i0}^2 - \rho_2 v_2 u_{i0}^2 - u_{r0}\rho_1 v_1 u_{i0} u_{r0} - \rho_2 v_2 u_{i0} u_{r0} + u_{r0}\rho_1 v_1 u_{i0} u_{r0} - \rho_2 v_2 u_{i0} u_{r0} - \rho_1 v_1 u_{r0}^2 - \rho_2 v_2 u_{r0}^2 = 0$
  - $u_{i0}(\rho_1 v_1 u_{i0} - \rho_2 v_2 u_{i0} - \rho_1 v_1 u_{r0} - \rho_2 v_2 u_{r0}) + u_{r0}(\rho_1 v_1 u_{i0} - \rho_2 v_2 u_{i0} - \rho_1 v_1 u_{r0} - \rho_2 v_2 u_{r0}) = 0$
  - $[u_{i0} + u_{r0}] \cdot [(\rho_1 v_1 - \rho_2 v_2) u_{i0} - (\rho_1 v_1 + \rho_2 v_2) u_{r0}] = 0$ (2.9.12)



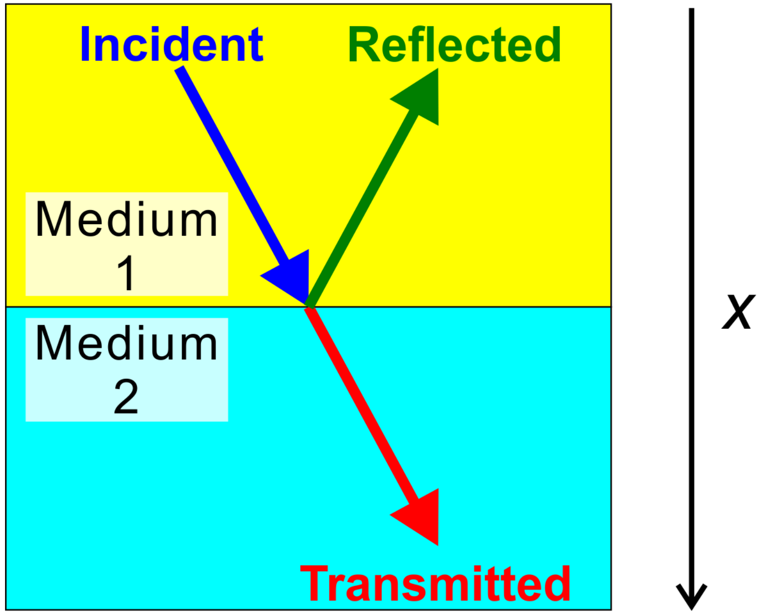
Reflection and transmission -5

- The first solution:
  - $u_{i0} + u_{r0} = 0$ (2.9.13)
    - Complete reflection
  - ✓ $u_{r0} = -u_{i0}$
    - The reflected waves have the same magnitudes of amplitude as, but an opposite phase to the incident wave.
      - $u_{t0} = u_{i0} + u_{r0} = 0$ (2.9.14)
      - No transmitted wave



Reflection and transmission -6

- The second solutions:
  - $(\rho_1 v_1 - \rho_2 v_2)u_{i0} - (\rho_1 v_1 + \rho_2 v_2)u_{r0} = 0$
    - ✓ Both reflection and transmission
    - ✓ $u_{r0} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} u_{i0}$ (2.9.15)
      - The amplitude of the reflected wave is proportional to the difference of ρv between two media, $\rho_1 v_1 - \rho_2 v_2$
      - Anti-phase reflection if $\rho_1 v_1 < \rho_2 v_2$
        - Medium 2 is stiffer and heavier, as is the “fixed end”
    - ✓ $u_{t0} = u_{i0} + \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} u_{i0} = \frac{2\rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2} u_{i0}$ (2.9.16)
      - The incident and transmitted waves gave the same phases because $\rho_1, v_1, \rho_2, v_2 > 0$
      - If $\rho_1 v_1 = \rho_2 v_2$, $u_{t0} = u_{i0}$: 100% transmission



Acoustic impedance

- ρv: an essential parameter for reflectivity and transmissivity
  - **Acoustic impedance**, $Z_S = \rho v$

Why is Z_S called "impedance"?

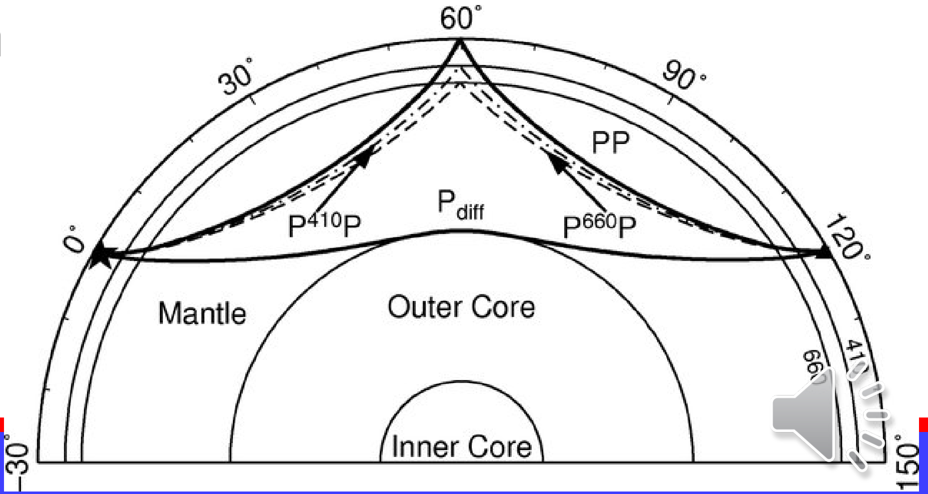
- In electricity: impedance = voltage / current
  - The ratio of the driving force to its results
- Acoustic impedance Z_S: the ratio of the stress to the particle velocity
- A small element of medium move by the stress.
  - Particle velocity:
    - ✓ $v_{particle} = \frac{\partial u}{\partial t} = \omega u_0 \sin(kx - \omega t)$ (2.9.17)
      - Wave function: $u = u_0 \cos(kx - \omega t)$
  - Stress:
    - ✓ $\sigma = E\varepsilon = E\frac{\partial u}{\partial x} = -Eku_0 \sin(kx - \omega t)$ (2.9.18)
      - $Z_S = \left|\frac{\sigma}{v_{particle}}\right| = \frac{Eku_0}{\omega u_0} = \frac{E}{\rho}\frac{k}{\omega}\rho = v_{phase}^2 \frac{1}{v_{phase}}\rho = \rho v_{phase}$ (2.9.19)

Reflectivity at the mantle discontinuities

- **P-wave** acoustic impedance of contrasts at **410-km** discontinuity
  - $Z_{S,P,410-} = 3.54\,\mathrm{g/cm^3} \times 8.91\,\mathrm{km/sec} = 3.15 \times 10^7\,\mathrm{kg/m^2sec}$
  - $Z_{S,P,410+} = 3.72\,\mathrm{g/cm^3} \times 9.13\,\mathrm{km/sec} = 3.40 \times 10^7\,\mathrm{kg/m^2sec}$
  - Reflectivity: $R_{P,400} = (3.15 \times 10^7 - 3.40 \times 10^7)/(3.40 \times 10^7 + 3.15 \times 10^7) = -3.7\%$
    - ✓ Overturn
- **P-wave** acoustic impedance of contrasts at **660-km** discontinuity
  - $Z_{S,P,660-} = 3.99\,\mathrm{g/cm^3} \times 10.27\,\mathrm{km/sec} = 4.10 \times 10^7\,\mathrm{kg/m^2sec}$
  - $Z_{S,P,660+} = 4.38\,\mathrm{g/cm^3} \times 10.75\,\mathrm{km/sec} = 4.71 \times 10^7\,\mathrm{kg/m^2sec}$
  - Reflectivity: $R_{P,400} = (4.10 \times 10^7 - 4.71 \times 10^7)/(4.71 \times 10^7 + 4.10 \times 10^7) = -6.9\%$
    - ✓ D660 is a stronger discontinuity than D410

Phase shift and overturn

- No phase shift by reflection and transmission
  - Reflectivity and transmissivity: real number
- Overturn
  - Incident from a low Z_S medium to high Z_S medium
    - ✓ Negative reflectivity $\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} < 0$: overturn
      - From the shallower to the deeper in many cases
  - Incident from a high Z_S medium to low Z_S medium
    - ✓ Positive reflectivity $\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} < 0$: no overturn
      - Underside reflection



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