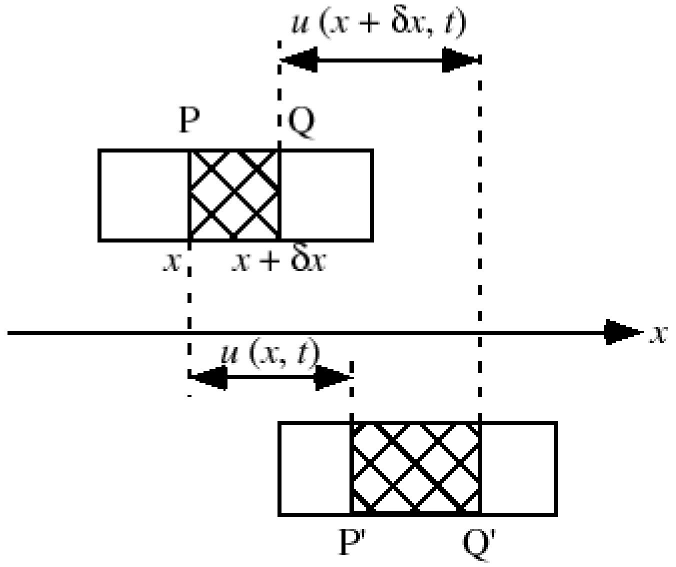
Mineral Physics I
Chapter 2. Elasticity
Section 8. Elastic wave velocity

Tomoo Katsura,

Bayerisches Geoinstitut, University of Bayreuth, Germany

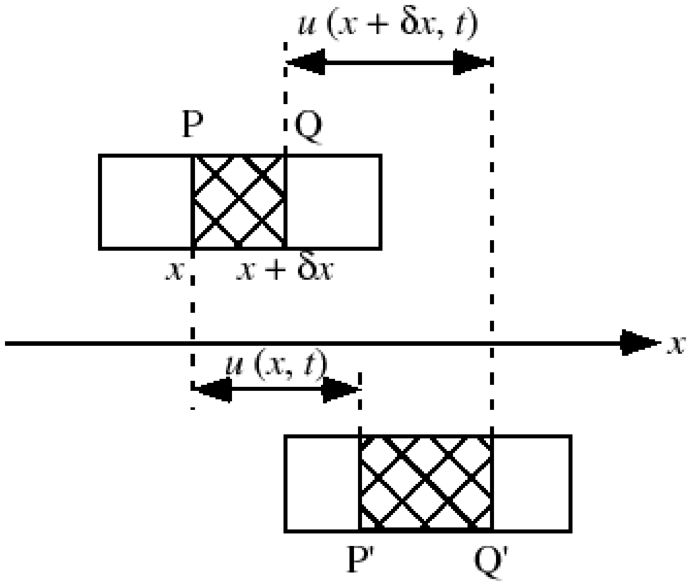
1D elastic wave velocity -1

- 1D infinitesimal volume element: $[x, x + \delta x]$
  - $P(x)$, $Q(x + \delta x)$
  - Mass of the volume element:
    - ✓ $m = \rho(x + \delta x - x) = \rho\delta x$ (2.8.1)
      - ρ: one-dimensional density
- Displacement of this element at time t by a wave
  - $[u(x,t), u(x + \delta x, t)]$
    - ✓ P (x)→ P'$\left(x + u(x,t)\right)$
    - ✓ Q $(x + \delta x)$→Q' $\left(x + \delta x + u(x + \delta x, t)\right)$



1D elastic wave velocity -2

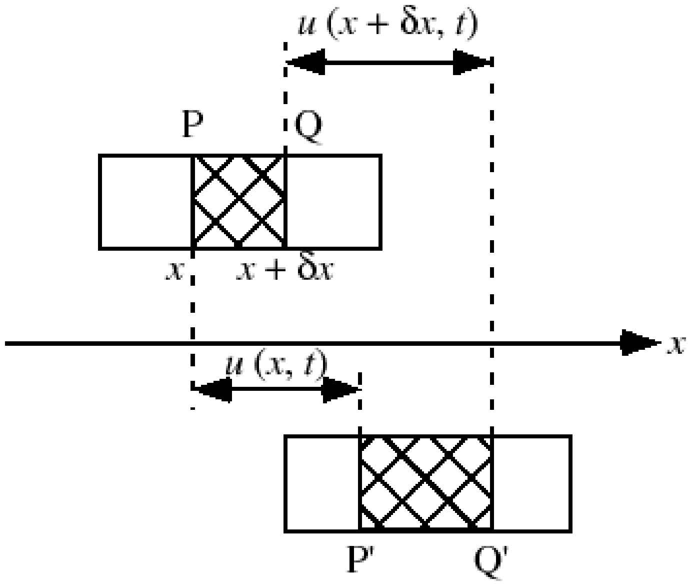
- Forces to the infinitesimal volume element under strain $[x + u(x,t) \quad x + \delta x + u(x+\delta x, t)]$ according to the Hooke's law.
  - On the minus side $(P'(x + u(x,t)))$:
    - ✓ $\sigma(x,t) = E\frac{\partial u(x,t)}{\partial x}$ (2.8.2)
      - E: elastic constant
  - On the plus side $(Q'(x + \delta x + u(x+\delta x,t)))$:
    - ✓ $\sigma(x+\delta x,t) = E\frac{\partial u(x+\delta x,t)}{\partial x}$ (2.8.3)



- The net force, F, to this element:
  - $F = \sigma(x+\delta x,t) - \sigma(x,t) = E\frac{\partial u(x+\delta x,t)}{\partial x} - E\frac{\partial u(x,t)}{\partial x} = E\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\delta x = E\frac{\partial^2 u}{\partial x^2}\delta x$ (2.8.4)

1D elastic wave velocity -2

- Forces to the infinitesimal volume element under strain $[x + u(x,t) \quad x + \delta x + u(x + \delta x, t)]$ according to the Hooke's law.
  - On the minus side $(P'\left(x + u(x,t)\right))$:
    - ✓ $\sigma(x,t) = E\dfrac{\partial u(x,t)}{\partial x}$ (2.8.2)
      - E: elastic constant
  - On the plus side $(Q'\left(x + \delta x + u(x + \delta x, t)\right))$:
    - ✓ $\sigma(x + \delta x, t) = E\dfrac{\partial u(x+\delta x,t)}{\partial x}$ (2.8.3)



- The net force, F, to this element:
  - $F = \sigma(x + \delta x, t) - \sigma(x,t) = E\dfrac{\partial u(x+\delta x,t)}{\partial x} - E\dfrac{\partial u(x,t)}{\partial x} = E\dfrac{\partial}{\partial x}\dfrac{\partial u}{\partial x}\delta x = E\dfrac{\partial^2 u}{\partial x^2}\delta x$ (2.8.4)

1D elastic wave velocity -3

- The equation of motion: $F = m\frac{d^2x}{dt^2}$
  - $E\frac{\partial^2 u}{\partial x^2}\delta x = F = m\frac{\partial^2 u}{\partial t^2}$ (2.8.5)
- The force and mass in (2.8.5) are substituted by (2.8.4) $F = E\frac{\partial^2 u}{\partial x^2}\delta x$ and (2.8.1) $m = \rho\delta x$
  - $E\left(\frac{\partial^2 u}{\partial x^2}\right)\delta x = \rho\delta x\left(\frac{\partial^2 u}{\partial t^2}\right)$
  - $\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$ (2.8.6)
    - ✓ The wave equation
- The solution of the wave equation:
  - $u = u_0 \exp[ik(x - vt)]$ (2.8.7)

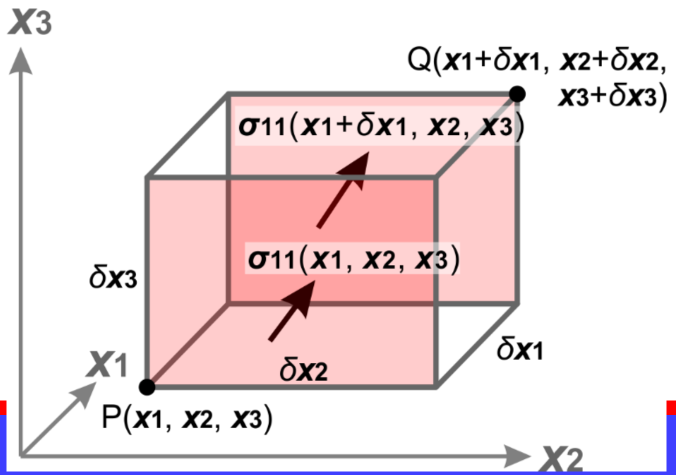
1D elastic wave velocity -4

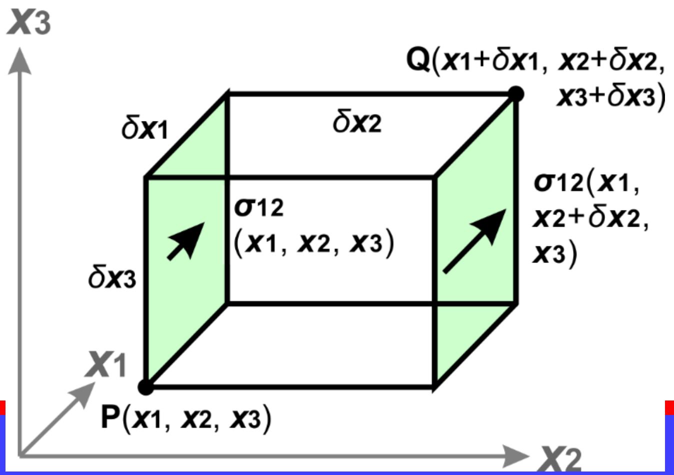
- The solution (2.8.7) $u = u_0 \exp[ik(x - vt)]$ is substituted into the wave equation (2.8.6) $\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$
  - Left side: $\frac{\partial^2 u}{\partial t^2} = -k^2 v^2 u_0 \exp[ik(x - vt)]$
  - Right side: $\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{E}{\rho}\{-k^2 u_0 \exp[ik(x - vt)]\}$
  - By equating them, $k^2 v^2 u_0 = \frac{E}{\rho}(k^2 u_0)$
  - $v = \sqrt{\frac{E}{\rho}}$ (2.8.8)
    - ✓ The velocity is proportional to the square root of the elastic constant and inversely to the density

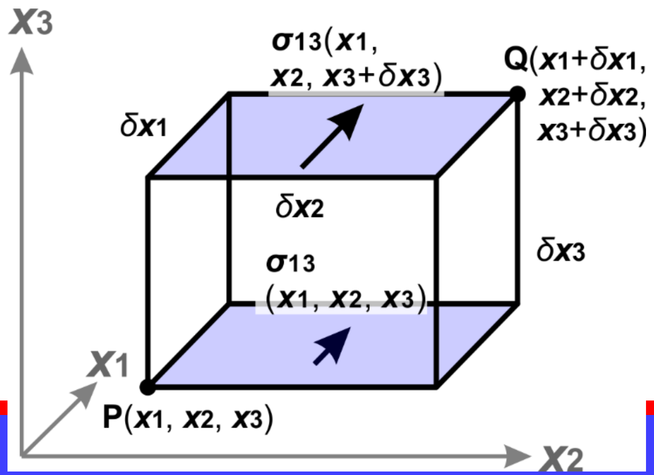
3D elastic wave velocities -1

- A 3D infinitesimal volume element $\delta x_1 \times \delta x_2 \times \delta x_3$
- The elastic forces acting on this element in the x_1 direction:
  - $F_{x_1} = \left(\frac{\partial \sigma_{11}}{\partial x_1}\delta x_1\right)\delta x_2 \delta x_3 + \left(\frac{\partial \sigma_{12}}{\partial x_2}\delta x_2\right)\delta x_1 \delta x_3 + \left(\frac{\partial \sigma_{13}}{\partial x_3}\delta x_3\right)\delta x_1 \delta x_2$

$$= \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}\right)\delta x_1 \delta x_2 \delta x_3 \qquad (2.8.9)$$







3D elastic wave velocities -1

- The equations of motion
  - With combining (2.8.9), the equation of motion in the x_1 direction: $F_{x_1} = m\frac{\partial^2 x_1}{\partial t^2}$
    - The mass of volume element $m = \rho\delta x_1\delta x_2\delta x_3$
  - ✓ $\left(\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3}\right)\delta x_1\delta x_2\delta x_3 = \rho\delta x_1\delta x_2\delta x_3\frac{\partial^2 u_1}{\partial t^2}$
  - ✓ $\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3}$ (2.8.10a)
  - Similarly in the x_2 and x_3 directions,
  - ✓ $\rho\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3}$ (2.8.10b)
  - ✓ $\rho\frac{\partial^2 u_3}{\partial t^2} = \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3}$ (2.8.10c)

From Section 2.5
Lame's constants -2

- $\sigma_{ij} = \lambda\delta_{ij}\sum_k \varepsilon_{kk} + 2\mu\varepsilon_{ij}$ (2.5.29)

- $$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ 2\varepsilon_4\\ 2\varepsilon_5\\ 2\varepsilon_6\end{bmatrix} \quad (2.5.31)$$

  - ✓ Satisfying $C_{11} = C_{12} + 2C_{44}$ (2.5.32)
    - $C_{11} = \lambda + 2\mu$ (2.5.33)
    - $C_{12} = \lambda$ (2.5.34)
    - $C_{44} = \mu$ (2.5.35)

- μ: rigidity, often expressed by G

3D elastic wave velocities -2

- The generalized Hooke's law of isotropic solids (2.5.31) using the Lame's constants,
  - $\sigma_1 = \sigma_{11} = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}$ (2.8.11a)
  - $\sigma_2 = \sigma_{22} = \lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3 = \lambda\frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}$ (2.8.11b)
  - $\sigma_3 = \sigma_{33} = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2\mu)\varepsilon_3 = \lambda\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3}$ (2.8.11c)
  - $\sigma_4 = \sigma_{23} = \sigma_{32} = 2\mu\varepsilon_4 = \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$ (2.8.11d)
  - $\sigma_5 = \sigma_{31} = \sigma_{13} = 2\mu\varepsilon_5 = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$ (2.8.11e)
  - $\sigma_6 = \sigma_{12} = \sigma_{21} = 2\mu\varepsilon_6 = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$ (2.8.11f)

3D elastic wave velocities -3

- Substituting (2.8.11a) $\sigma_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}$, (2.5.11f) $\sigma_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$, (2.8.11e) $\sigma_{13} = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$, into the equation of motion in the x_1 direction (2.8.10a)
- $\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$

$$= \frac{\partial}{\partial x_1}\left[(\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}\right] + \frac{\partial}{\partial x_2}\left[\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_3}\left[\mu\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)\right]$$

$$= (\lambda + \mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\lambda\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\lambda\frac{\partial u_3}{\partial x_3} + \frac{\partial}{\partial x_2}\mu\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_2}\mu\frac{\partial u_2}{\partial x_1} + \frac{\partial}{\partial x_3}\mu\frac{\partial u_3}{\partial x_1} + \frac{\partial}{\partial x_3}\mu\frac{\partial u_1}{\partial x_3}$$

$$= \mu\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + (\lambda + \mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \mu\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \mu\frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}$$

$$= \mu\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right) + (\lambda + \mu)\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right) \qquad (2.8.12)$$

3D elastic wave velocities -3

- Substituting (2.8.11a) $\sigma_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}$, (2.5.11f) $\sigma_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$, (2.8.11e) $\sigma_{13} = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$, into (2.8.10a)
- $\rho\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$

$$= \frac{\partial}{\partial x_1}\left[(\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}\right] + \frac{\partial}{\partial x_2}\left[\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_3}\left[\mu\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)\right]$$

$$= (\lambda + \mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\lambda\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\lambda\frac{\partial u_3}{\partial x_3} + \frac{\partial}{\partial x_2}\mu\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_2}\mu\frac{\partial u_2}{\partial x_1} + \frac{\partial}{\partial x_3}\mu\frac{\partial u_3}{\partial x_1} + \frac{\partial}{\partial x_3}\mu\frac{\partial u_1}{\partial x_3}$$

$$= \mu\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + (\lambda + \mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \mu\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \mu\frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}$$

$$= \mu\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right) + (\lambda + \mu)\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right) \quad (2.8.12)$$

3D elastic wave velocities -4
Compressional wave velocity

- $\rho \frac{\partial^2 u_1}{\partial t^2} = \mu\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right) + (\lambda + \mu)\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right)$ (2.8.12)
- $\rho \frac{\partial^2 u_1}{\partial t^2} = \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + (\lambda + \mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$

  $= \mu \nabla^2 u_1 + (\lambda + \mu)\frac{\partial}{\partial x_1}\mathrm{div}(\boldsymbol{u})$ (2.8.13a)

- Similarly
  - $\rho \frac{\partial^2 u_2}{\partial t^2} = \mu \nabla^2 u_2 + (\lambda + \mu)\frac{\partial}{\partial x_2}\mathrm{div}(\boldsymbol{u})$ (2.8.13b)
  - $\rho \frac{\partial^2 u_3}{\partial t^2} = \mu \nabla^2 u_3 + (\lambda + \mu)\frac{\partial}{\partial x_3}\mathrm{div}(\boldsymbol{u})$ (2.8.13c)
- By combining the 3 components (2.8.12a), (2.8.12b) and (2.8.12c):
  - $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu)\mathrm{grad}(\mathrm{div}(\boldsymbol{u})) = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu)\nabla^2 \boldsymbol{u}$
  - $\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + 2\mu)\nabla^2 \boldsymbol{u}$ (2.8.14)

3D elastic wave velocities -5
Compressional wave velocity

- The wave with velocity
  - $\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \frac{\lambda+2\mu}{\rho} \nabla^2 \boldsymbol{u}$ (2.8.13)
    - ✓ $\frac{\partial^2 u_1}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right)$
    - ✓ $\frac{\partial^2 u_2}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right)$
    - ✓ $\frac{\partial^2 u_3}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right)$
  - This is an equation of wave. The wave velocity is:
    - $v_p = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\left(K + \frac{4}{3}G\right)/\rho}$ (2.8.14) $K = \lambda + \frac{2}{3}\mu$ (2.6.7)
    - ✓ Compressional wave

3D elastic wave velocities -6
Shear wave velocity

- The x_3 derivative of (2.8.13b) $\rho \frac{\partial^2 u_2}{\partial t^2} = \mu \nabla^2 u_2 + (\lambda + \mu) \frac{\partial}{\partial x_2} \mathrm{div}(\boldsymbol{u})$ (the x_2 component)
  - $$\rho \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial x_3} u_2 = \mu \nabla^2 \frac{\partial u_2}{\partial x_3} + \cancel{(\lambda + \mu) \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} \mathrm{div}(\boldsymbol{u})} \quad (2.8.15)$$
- The x_2 derivative of (2.8.13c) $\rho \frac{\partial^2 u_3}{\partial t^2} = \mu \nabla^2 u_3 + (\lambda + \mu) \frac{\partial}{\partial x_3} \mathrm{div}(\boldsymbol{u})$ (the x_3 component)
  - $$\rho \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial x_2} u_3 = \mu \nabla^2 \frac{\partial u_3}{\partial x_2} + (\lambda + \mu) \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} \mathrm{div}(\boldsymbol{u})$$
    $$= \mu \nabla^2 \frac{\partial u_3}{\partial x_2} + \cancel{(\lambda + \mu) \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_2} \mathrm{div}(\boldsymbol{u})} \quad (2.8.16)$$
- Subtracting (2.8.15) from (2.8.16), we have
  - $$\rho \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right) = \mu \nabla^2 \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \right) \quad (2.8.17a)$$

3D elastic wave velocities -6
Shear wave velocity

- Similarly to (2.8.17a), $\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) = \mu \nabla^2 \left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right)$
  - $\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) = \mu \nabla^2 \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right)$ (2.8.17b)
  - $\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = \mu \nabla^2 \left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$ (2.8.17c)
- The definition of $\mathrm{rot}(\boldsymbol{a}) \equiv \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \quad \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \quad \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right)$ (2.1.29)
  - $\frac{\partial^2}{\partial t^2}\mathrm{rot}(\boldsymbol{u}) = \frac{\mu}{\rho}\nabla^2 \mathrm{rot}(\boldsymbol{u})$ (2.8.18)
    - ✓ An equation of wave $y = \mathrm{rot}(\boldsymbol{u})$ with the velocity

      $v_s = \sqrt{\mu/\rho} = \sqrt{G/\rho}$ (2.8.19)
    - ✓ **Shear wave**, Twist

Bulk sound velocity

- Liquid or gas: zero rigidity
  - No shear wave
  - Compressional wave only
    - ✓ Wave velocity:

$$v_\Phi = \sqrt{\left(K + \frac{4}{3}0\right)/\rho} = \sqrt{K/\rho} \quad (2.8.27)$$

- Apply this formula to solids => **bulk sound velocity**
  - although no “bulk sound” exists
  - Rigidity is more difficult to determine than bulk modulus
  - v_Φ can be calculated from seismic observations of v_P and v_S
    - ✓ $v_\Phi^2 = v_\mathrm{P}^2 - \frac{4}{3} v_\mathrm{S}^2$ (2.8.28)

Mineral Physics I

Chapter 2. Elasticity

Section 8. Elastic wave velocity

End