Mineral Physics I
Chapter 2. Elasticity
Section 6. Frequently used elastic constants

Tomoo Katsura,

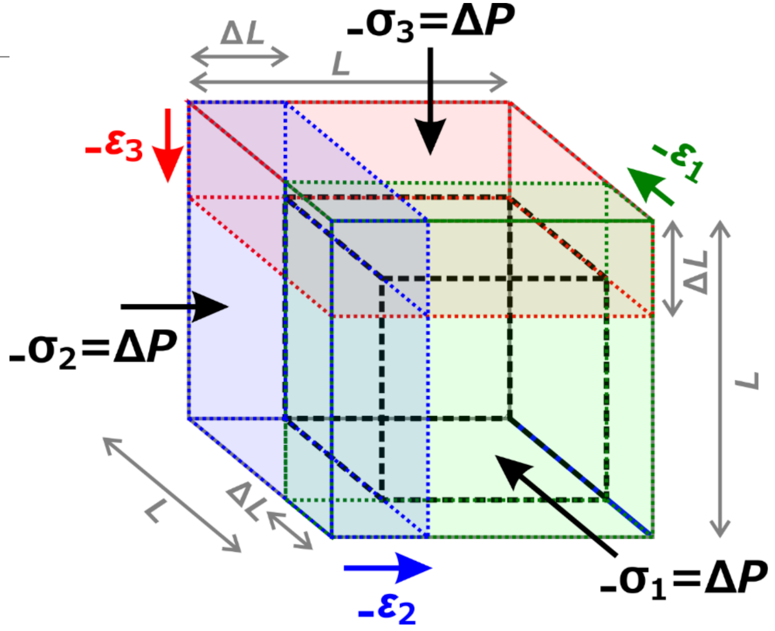
Bayerisches Geoinstitut, University of Bayreuth, Germany

Bulk modulus -1

- Increase in hydrostatic pressure by volume decrease:
  - $K \equiv \Delta P/(-\Delta V/V)$ (2.6.1)
- No shear strain but normal strains with the same magnitude only:
  - $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ (2.6.2)
  - $\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0$ (2.6.3)
- $$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_1 \\ \varepsilon_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ (2.6.4)
- $\sigma_{1,2,3} = 3\lambda\varepsilon_1 + 2\mu\varepsilon_1 = (3\lambda + 2\mu)\varepsilon_1$ (2.6.5)

Bulk modulus -2

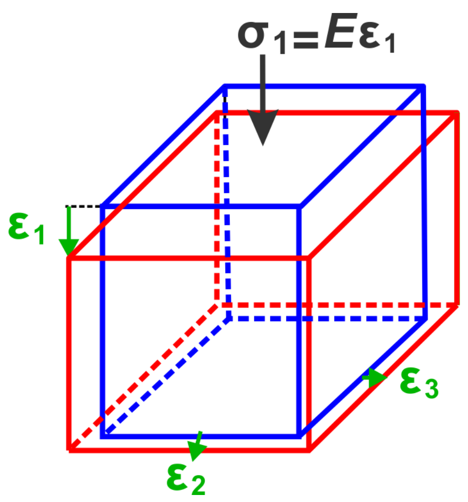
- ❑ $\sigma_{1,2,3} = (3\lambda + 2\mu)\varepsilon_1$ (2.6.5)
- ❑ Pressure: the averaged normal stress
  - ➢ $\Delta P = -\frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -(3\lambda + 2\mu)\varepsilon_1$ (2.6.6)
- ❑ Volume change: sum of normal strains
  - ➢ $\Delta V/V = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\varepsilon_1$ (2.2.13)
    - ✓ $\varepsilon_1 = \Delta L/L$
    - ✓ $V = L^3$
    - ✓ $\Delta V = 3\Delta L \cdot L^2$
    - ✓ $\Delta V/V = (3\Delta L \cdot L^2/L^3) = 3\,\Delta L/L = 3\varepsilon_1$
- ❑ The bulk modulus:
  - ➢ $K \equiv -\frac{\Delta P}{(\Delta V/V)} = -\frac{-(3\lambda+2\mu)\varepsilon_1}{3\varepsilon_1} = \lambda + \frac{2}{3}\mu$ (2.6.7)



The volume decrease in one side, by length shortening, ΔL :
$\Delta L \cdot L^2$
The relative volume change:
$\Delta L \cdot L^2/L^3 = \Delta L/L = \varepsilon_1$

Young's modulus -1

- ❑ Young's modulus, E: the ratio of the uniaxial stress to the strain in the direction of the uniaxial stress without other stresses:
  - ➢ $\sigma_1 = E\varepsilon_1$ where $\sigma_{i\neq 1} = 0$ (2.6.8)
- ❑ The relations of the Young's modulus with the Lame's constants in the case of isotropic materials
  - ➢ $$\begin{bmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix} \quad (2.6.9)$$



Young's modulus -2

- From Eq. (2.6.9), we have
  - $\sigma_1 = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$ (2.6.10)
  - $\sigma_2 = 0 = \lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3$ (2.6.11)
  - $\sigma_3 = 0 = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2\mu)\varepsilon_3$ (2.6.12)
- By combining Eq. (2.6.8) $\sigma_1 = E\varepsilon_1$ and (2.6.10), we have
  - $E\varepsilon_1 = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$
  - $(\lambda + 2\mu - E)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 = 0$ (2.6.13)
- By combining Eq. (2.6.11) and (2.6.13), we have
  - $\lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3 = (\lambda + 2\mu - E)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$
  - $\varepsilon_2 = \frac{2\mu - E}{2\mu}\varepsilon_1$ (2.6.14)

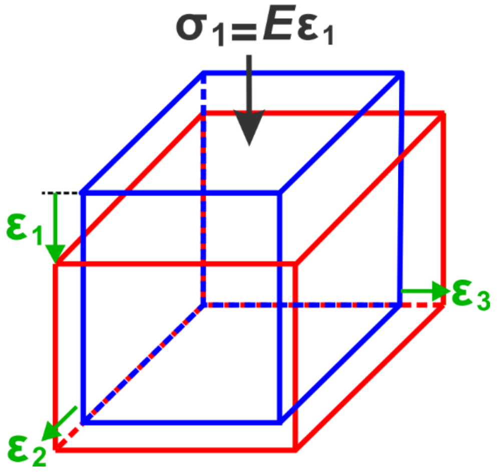
Young's modulus -3

- Similarly,
  - $\varepsilon_3 = \frac{2\mu - E}{2\mu}\varepsilon_1$ (2.6.15)
- By substituting (2.6.14) $\varepsilon_2 = \frac{2\mu - E}{2\mu}\varepsilon_1$ and (2.6.15) into (2.6.13) $(\lambda + 2\mu - E)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 = 0$, we have
  - $(\lambda + 2\mu - E)\varepsilon_1 + \lambda\frac{2\mu - E}{2\mu}\varepsilon_1 + \lambda\frac{2\mu - E}{2\mu}\varepsilon_1 = 0$
  - $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ (2.6.16)
- The Young's modulus E can be expressed using the bulk modulus K and rigidity $G = \mu$:
  - $E = 9KG/(3K + G)$ (2.6.17)

Poisson's ratio -1

- The uniaxial stress increases the dimension of the body normal to the uniaxial stress

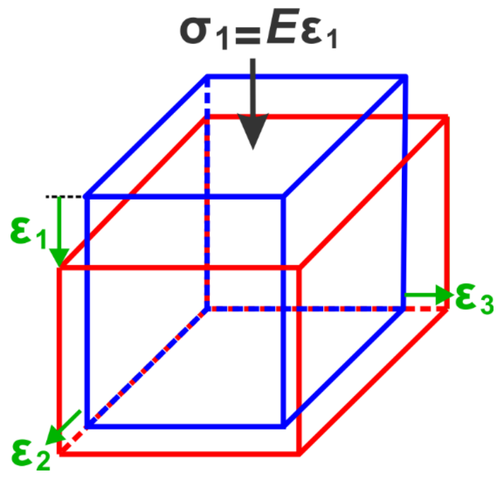
- Poisson's ratio: the negative ratio of the transverse strain to the axial strain
  - $\nu \equiv -\varepsilon_2/\varepsilon_1 = -\varepsilon_3/\varepsilon_1$ (2.6.17)
    - ✓ where $\sigma_1 = E\varepsilon_1$, and $\sigma_{2,3} = 0$



- The ν can be expressed using the E and μ from (2.6.14) $\varepsilon_2 = \{(2\mu - E)/2\mu\}\varepsilon_1$
  - $\nu = -\dfrac{\varepsilon_1(2\mu - E)}{2\mu}\Big/\varepsilon_1 = \dfrac{E - 2\mu}{2\mu}$ (2.6.18)

Poisson's ratio -2

- From (2.6.16) $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$, ν can be expressed using λ and μ :
  - $\nu = \lambda/(2\lambda + 2\mu)$ (2.6.19)
- The ν can be expressed using K and $G = \mu$ from (2.6.7), $K = \lambda + (2/3)\mu$
  - $\nu = (3K - 2G)/(6K + 2G)$ (2.6.20)
- $-1 < \nu < +0.5$ (2.6.21)
  - $\nu \to -1$ when $G \gg K$
  - $\nu \to 0.5$ when $\lambda \gg \mu$



K vs T

Al_2O_3: very high

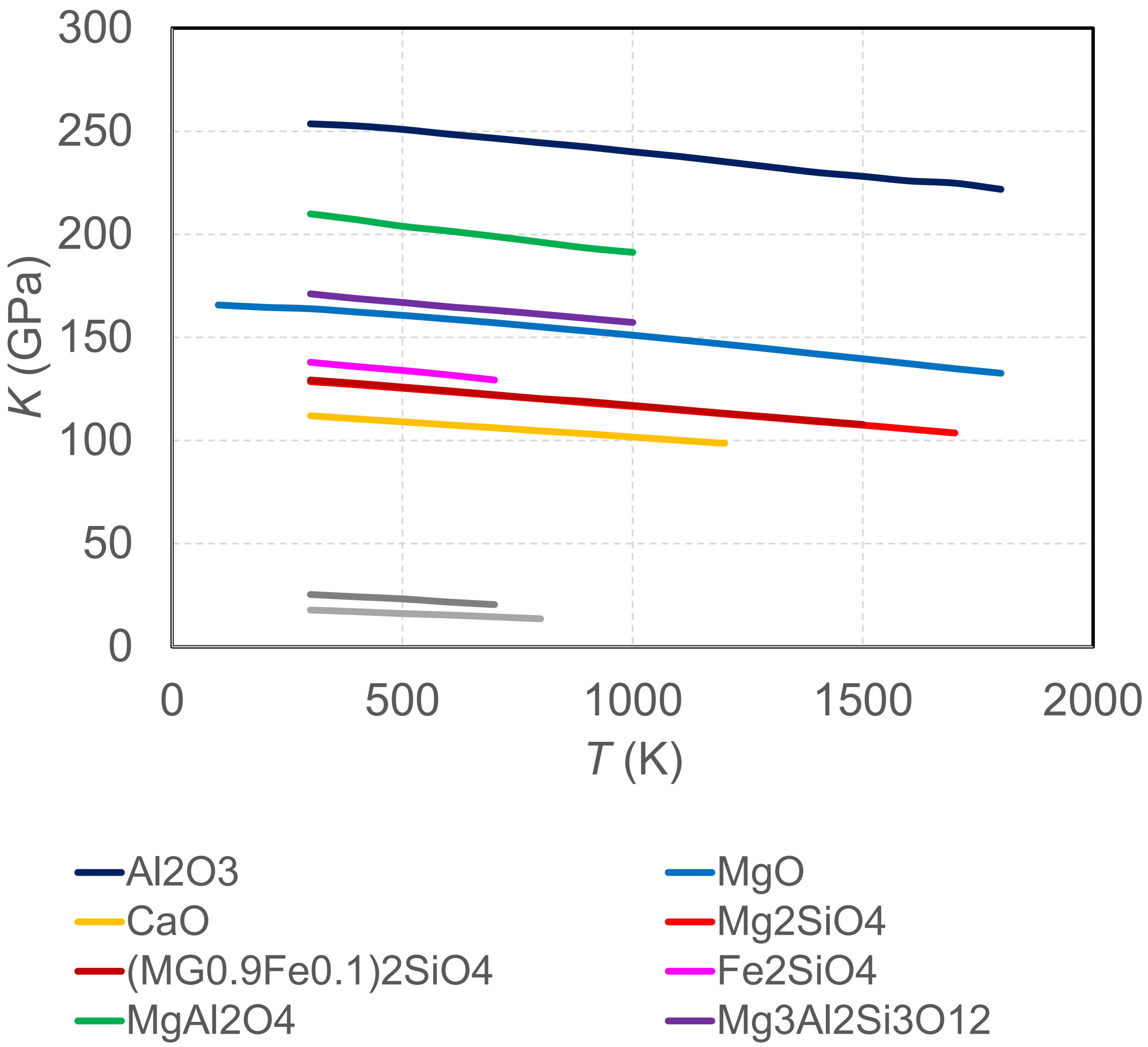
Alkali halide: very low

Olivine: relatively low for oxide

Garnet: higher than olivine

Spinel: high

Alkali Earth oxide: relatively low, especially heavier one.



G vs *T*

Al2O3: very high

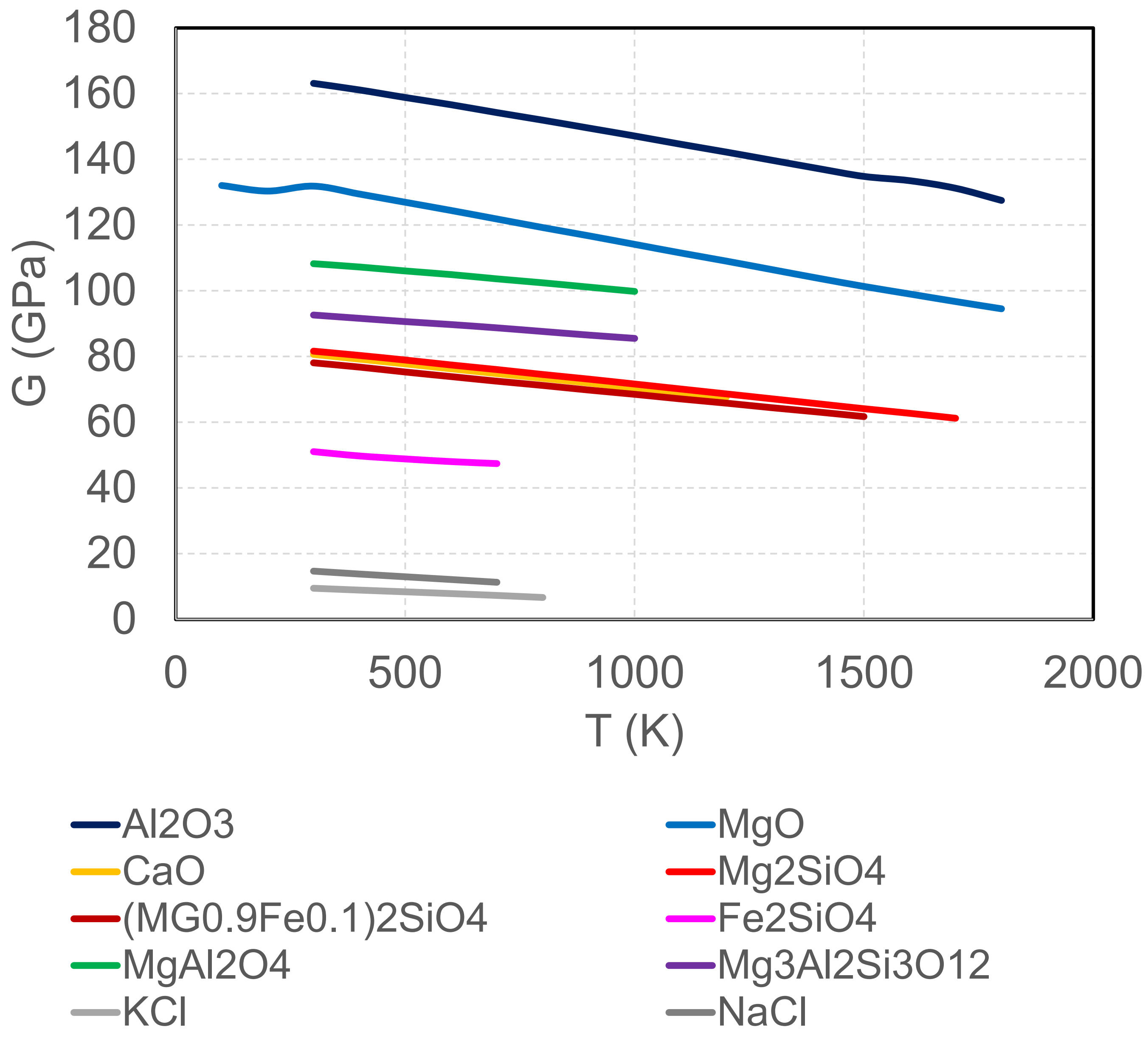
Alkali halide: very low

Olivine: relatively low for oxide

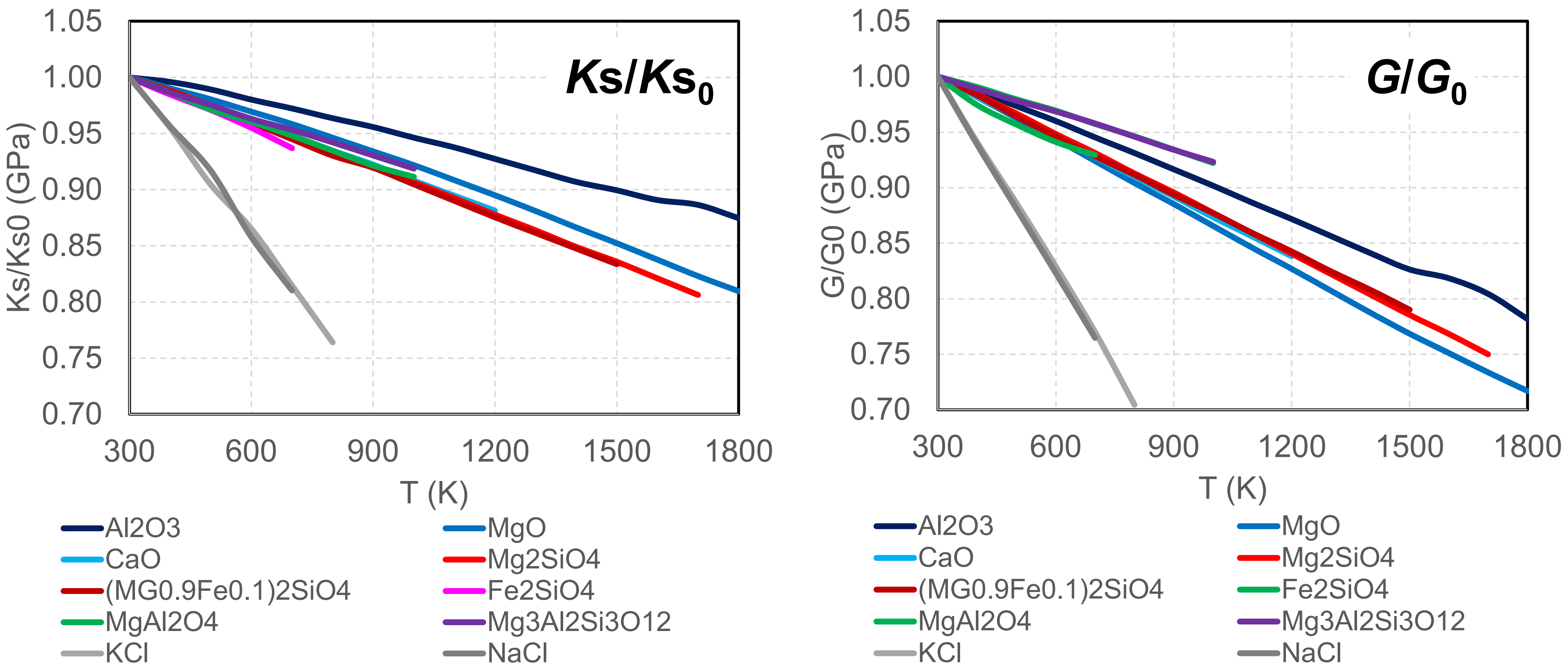
Garnet: higher than olivine

Spinel: relatively high

Alkali Earth oxide: relatively high.



Ks/Ks_0 & G/G_0 vs T



G decreases more rapidly with *T* than *Ks*

Mineral Physics I
Chapter 1. Thermodynamic properties
Section 6. Frequently used elastic constants

End