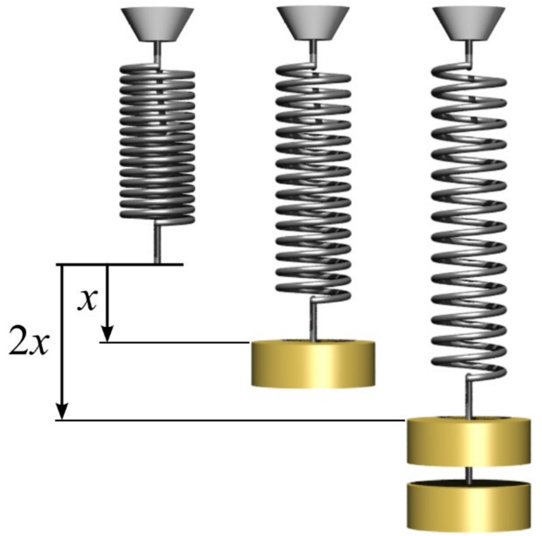
Mineral Physics I
Chapter 2. Elasticity
Section 4. Generalized Hooke's law

Tomoo Katsura,

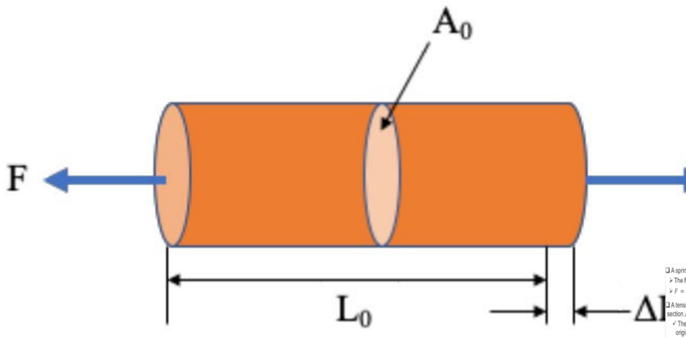
Bayerisches Geoinstitut, University of Bayreuth, Germany

Hooke's law

- A spring is extended by the length, x with the force F
  - The force F : proportional to x (Hooke's law)
  - $F = kx$ (2.4.1)



- A tensile test to a rod with length L_0 and cross section A_0.
  - ✓ The length of extension, ΔL: proportional to the original length L_0 → strain $\varepsilon = \Delta L / L_0$
  - ✓ The force for extension, F: proportional to the cross section A_0 → stress $\sigma = F / A_0$
  - ✓ Hooke's law
    - $\sigma = C\varepsilon$ (2.4.2)
  - ✓ Linear elasticity



Generalized Hooke's law

- Generalized Hooke's law
  - If a body is strained in some way as ε_{ij}, not only σ_{ij} but various kinds of stresses occur.
  - The stress σ_{ij} is generated not only by ε_{ij} but other strains could contribute to it as well.
  - $\sigma_{ij} = \sum_{kl} C_{ijkl}\varepsilon_{kl}\ (i, j, k, l = 1, 2, 3)$ (2.4.3)
    - ✓ $\sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}$
      - C_{ijkl}: Stiffness, elastic moduli or elastic constants
        - 4th-rank tensor

Matrix of generalized Hooke's law

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121} \\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221} \\
C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321} \\
C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321} \\
C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221} \\
C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121} \\
C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321} \\
C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221} \\
C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix}
\quad (2.4.4)
$$

- σ and ε: not in the ascending order, but (1, 1). (1, 2). (1, 3). (2, 1). (2, 2). (2, 3). (3, 1). (3, 2). (3, 3)
- C_{ijkl}: Stiffness, elastic moduli

Compliance

- The inversed relations between strains and stresses
  - $\varepsilon_{ij} = \sum_{kl} S_{ijkl}\,\sigma_{kl}$ (2.4.5)
    - ✓ S_{ijkl}: compliance
  - $[S_{ijkl}] = [C_{ijkl}]^{-1}$ (2.4.6)
  - $$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1132} & S_{1131} & S_{1113} & S_{1112} & S_{1121} \\ S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2232} & S_{2231} & S_{2213} & S_{2212} & S_{2221} \\ S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3332} & S_{3331} & S_{3313} & S_{3312} & S_{3321} \\ S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2332} & S_{2331} & S_{2313} & S_{2312} & S_{2321} \\ S_{3211} & S_{3222} & S_{3233} & S_{3223} & S_{3232} & S_{3231} & S_{3213} & S_{3212} & S_{3221} \\ S_{3111} & S_{3122} & S_{3133} & S_{3123} & S_{3132} & S_{3131} & S_{3113} & S_{3112} & S_{3121} \\ S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1332} & S_{1331} & S_{1313} & S_{1312} & S_{1321} \\ S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1232} & S_{1231} & S_{1213} & S_{1212} & S_{1221} \\ S_{2111} & S_{2122} & S_{2133} & S_{2123} & S_{2132} & S_{2131} & S_{2113} & S_{2112} & S_{2121} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} \quad (2.4.7)$$

Symmetricity of the elastic constant tensor -1

- The symmetricity of the elastic constant tensors,
  - ✓ $C_{ijkl} = C_{jikl} \leftarrow \sigma_{ij} = \sigma_{ji}$ (2.4.8)
  - ✓ $C_{ijkl} = C_{ijlk} \leftarrow \varepsilon_{kl} = \varepsilon_{lk}$ (2.4.9)
  - ✓ $C_{ijkl} = C_{klij} \leftarrow \sigma_{ij}/\varepsilon_{kl} = \sigma_{kl}/\varepsilon_{ij}$ (2.4.10)
  - ➢ $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} = C_{klij} = C_{lkij} = C_{klji} = C_{lkji}$ (2.4.11)
  - ✓ $9 \times 9 = 81$ components → 21 independent components

- Proof of (2.4.10)
  - ➢ The work by the strains is stored in the body as the strain energy
    - ✓ $U = \int \sigma_{11} \mathrm{d}\varepsilon_{11} + \int \sigma_{12} \mathrm{d}\varepsilon_{12} + \int \sigma_{13} \mathrm{d}\varepsilon_{13} + \int \sigma_{21} \mathrm{d}\varepsilon_{21} + \int \sigma_{22} \mathrm{d}\varepsilon_{22} + \int \sigma_{23} \mathrm{d}\varepsilon_{23} + \int \sigma_{31} \mathrm{d}\varepsilon_{31} + \int \sigma_{32} \mathrm{d}\varepsilon_{32} + \int \sigma_{33} \mathrm{d}\varepsilon_{33}$ (2.4.12)
    - ✓ On the combinations of the stresses and the strains in the same direction do work.

Symmetricity of the elastic constant tensor -1

- The symmetricity of the elastic constant tensors,
  - ✓ $C_{ijkl} = C_{jikl} \leftarrow \sigma_{ij} = \sigma_{ji}$ (2.4.8)
  - ✓ $C_{ijkl} = C_{ijlk} \leftarrow \varepsilon_{kl} = \varepsilon_{lk}$ (2.4.9)
  - ✓ $C_{ijkl} = C_{klij} \leftarrow \sigma_{ij}/\varepsilon_{kl} = \sigma_{kl}/\varepsilon_{ij}$ (2.4.10)
  - ➢ $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} = C_{klij} = C_{lkij} = C_{klji} = C_{lkji}$ (2.4.11)
  - ✓ $9 \times 9 = 81$ components → 21 independent components

- Proof of (2.4.10)
  - ➢ The work by the strains is stored in the body as the strain energy
  - ✓ $U = \int \sigma_{11} \mathrm{d}\varepsilon_{11} + \int \sigma_{12} \mathrm{d}\varepsilon_{12} + \int \sigma_{13} \mathrm{d}\varepsilon_{13} + \int \sigma_{21} \mathrm{d}\varepsilon_{21} + \int \sigma_{22} \mathrm{d}\varepsilon_{22} + \int \sigma_{23} \mathrm{d}\varepsilon_{23} + \int \sigma_{31} \mathrm{d}\varepsilon_{31} + \int \sigma_{32} \mathrm{d}\varepsilon_{32} + \int \sigma_{33} \mathrm{d}\varepsilon_{33}$ (2.4.12)
  - ✓ On the combinations of the stresses and the strains in the same direction do work.

Symmetricity of the elastic constant tensor -2

- The total differential of U:
  - ✓ $dU = \sigma_{11}d\varepsilon_{11} + \sigma_{12}d\varepsilon_{12} + \sigma_{13}d\varepsilon_{13} + \sigma_{21}d\varepsilon_{21} + \sigma_{22}d\varepsilon_{22} + \sigma_{23}d\varepsilon_{23} + \sigma_{31}d\varepsilon_{31} + \sigma_{32}d\varepsilon_{32} + \sigma_{33}d\varepsilon_{33}$ (2.4.13)
- Mathematically, the differential of U
  - ✓ $dU = \frac{\partial U}{\partial \varepsilon_{11}}d\varepsilon_{11} + \frac{\partial U}{\partial \varepsilon_{12}}d\varepsilon_{12} + \frac{\partial U}{\partial \varepsilon_{13}}d\varepsilon_{13} + \frac{\partial U}{\partial \varepsilon_{21}}d\varepsilon_{21} + \frac{\partial U}{\partial \varepsilon_{22}}d\varepsilon_{22} + \frac{\partial U}{\partial \varepsilon_{23}}d\varepsilon_{23} + \frac{\partial U}{\partial \varepsilon_{31}}d\varepsilon_{31} + \frac{\partial U}{\partial \varepsilon_{32}}d\varepsilon_{32} + \frac{\partial U}{\partial \varepsilon_{33}}d\varepsilon_{33}$ (2.4.14)
  - ✓ $\sigma_{11} = \frac{\partial U}{\partial \varepsilon_{11}}, \sigma_{12} = \frac{\partial U}{\partial \varepsilon_{12}}, \sigma_{13} = \frac{\partial U}{\partial \varepsilon_{13}}, \sigma_{21} = \frac{\partial U}{\partial \varepsilon_{21}}, \sigma_{22} = \frac{\partial U}{\partial \varepsilon_{22}}, \sigma_{23} = \frac{\partial U}{\partial \varepsilon_{23}}, \sigma_{31} = \frac{\partial U}{\partial \varepsilon_{31}}, \sigma_{32} = \frac{\partial U}{\partial \varepsilon_{32}}, \sigma_{33} = \frac{\partial U}{\partial \varepsilon_{33}}$ (2.4.15)

Symmetricity of the elastic constant tensor -3

- ✓ From the generalized Hooke's law (2.4.3)
  - $\frac{\partial U}{\partial \varepsilon_{ij}} = \sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}$ (2.4.16)
- ➢ Since all ε are independent, the other terms than ε_{kl} disappear by the differentiation with respect to ε_{kl}.
  - ✓ $\frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = C_{ijkl} \frac{\mathrm{d}\varepsilon_{kl}}{\mathrm{d}\varepsilon_{kl}} = C_{ijkl}$ (2.4.17)
    - The elastic constants; the second derivative of strain energy respect to the strains
- ➢ Since the order of the differential is exchangeable,
  - ✓ $\frac{\partial^2 U}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = \frac{\partial^2 U}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}}$
  - ✓ $C_{ijkl} = C_{klij}$ (2.4.10)

Symmetricity of the elastic constant tensor -4

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_{21}\end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321}\\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221}\\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{32}\\ \varepsilon_{31}\\ \varepsilon_{13}\\ \varepsilon_{12}\\ \varepsilon_{21}\end{bmatrix} \quad (2.4.4)$$

- $C_{ijkl} = C_{jikl}$ (because of the symmetricity of stresses $\sigma_{ij} = \sigma_{ji}$) (2.4.8)

Symmetricity of the elastic constant tensor -5

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_{21}\end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321}\\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221}\\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121}\end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{32}\\ \varepsilon_{31}\\ \varepsilon_{13}\\ \varepsilon_{12}\\ \varepsilon_{21}\end{bmatrix} \quad (2.4.4)$$

- $C_{ij\boldsymbol{kl}} = C_{ij\boldsymbol{lk}}$ (because of the symmetry of the strains $\varepsilon_{kl} = \varepsilon_{lk}$) (2.4.9)

Symmetricity of the elastic constant tensor -6

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_{21}\end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321}\\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221}\\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121} \end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{32}\\ \varepsilon_{31}\\ \varepsilon_{13}\\ \varepsilon_{12}\\ \varepsilon_{21}\end{bmatrix} \tag{2.4.4}$$

- $C_{ijkl} = C_{klij}$ (because of the symmetricity of stress and strain $\sigma_{ij}/\varepsilon_{kl} = \sigma_{kl}/\varepsilon_{ij}$) (2.4.10)

Symmetricity of the elastic constant tensor -7

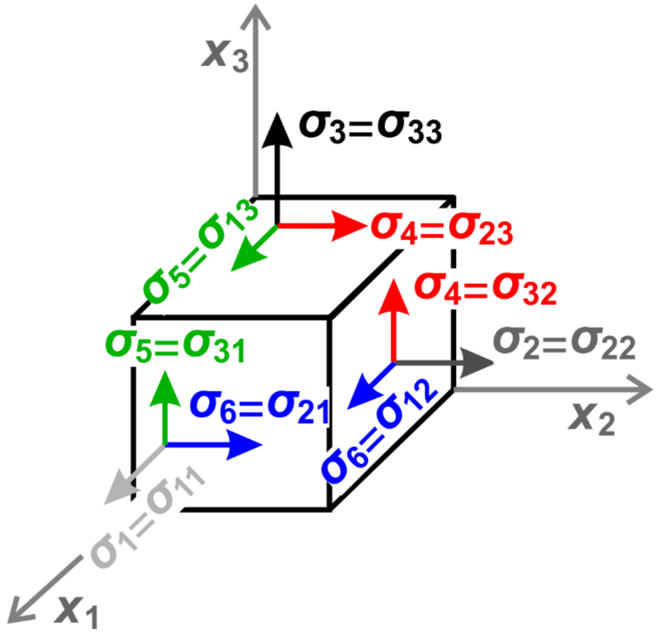
$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{32}\\\sigma_{31}\\\sigma_{13}\\\sigma_{12}\\\sigma_{21}\end{bmatrix}=\begin{bmatrix}C_{1111}&C_{1122}&C_{1133}&C_{1123}&C_{1132}&C_{1131}&C_{1113}&C_{1112}&C_{1121}\\C_{2211}&C_{2222}&C_{2233}&C_{2223}&C_{2232}&C_{2231}&C_{2213}&C_{2212}&C_{2221}\\C_{3311}&C_{3322}&C_{3333}&C_{3323}&C_{3332}&C_{3331}&C_{3313}&C_{3312}&C_{3321}\\C_{2311}&C_{2322}&C_{2333}&C_{2323}&C_{2332}&C_{2331}&C_{2313}&C_{2312}&C_{2321}\\C_{3211}&C_{3222}&C_{3233}&C_{3223}&C_{3232}&C_{3231}&C_{3213}&C_{3212}&C_{3221}\\C_{3111}&C_{3122}&C_{3133}&C_{3123}&C_{3132}&C_{3131}&C_{3113}&C_{3112}&C_{3121}\\C_{1311}&C_{1322}&C_{1333}&C_{1323}&C_{1332}&C_{1331}&C_{1313}&C_{1312}&C_{1321}\\C_{1211}&C_{1222}&C_{1233}&C_{1223}&C_{1232}&C_{1231}&C_{1213}&C_{1212}&C_{1221}\\C_{2111}&C_{2122}&C_{2133}&C_{2123}&C_{2132}&C_{2131}&C_{2113}&C_{2112}&C_{2121}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{32}\\\varepsilon_{31}\\\varepsilon_{13}\\\varepsilon_{12}\\\varepsilon_{21}\end{bmatrix} \qquad (2.4.4)$$

- $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} = C_{klij} = C_{lkij} = C_{klji} = C_{lkji}$ (2.4.11)
  - ✓ The same color: the same values
  - ✓ 81 components → 21 independent components

Voigt notation - stress

- Stress, strain: 2nd-rank tensor, stiffness, compliance: 4th-rank tensor ⇒ complex to describe
- Voigt notation
- 11 → 1, 22 → 2, 33 → 3, 23, 32 → 4, 13, 31 → 5, 12, 21 → 6
  - Stress

$$\checkmark \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{1} & \sigma_{6} & \sigma_{5} \\ \sigma_{6} & \sigma_{2} & \sigma_{4} \\ \sigma_{5} & \sigma_{4} & \sigma_{3} \end{bmatrix} \quad (2.4.18)$$

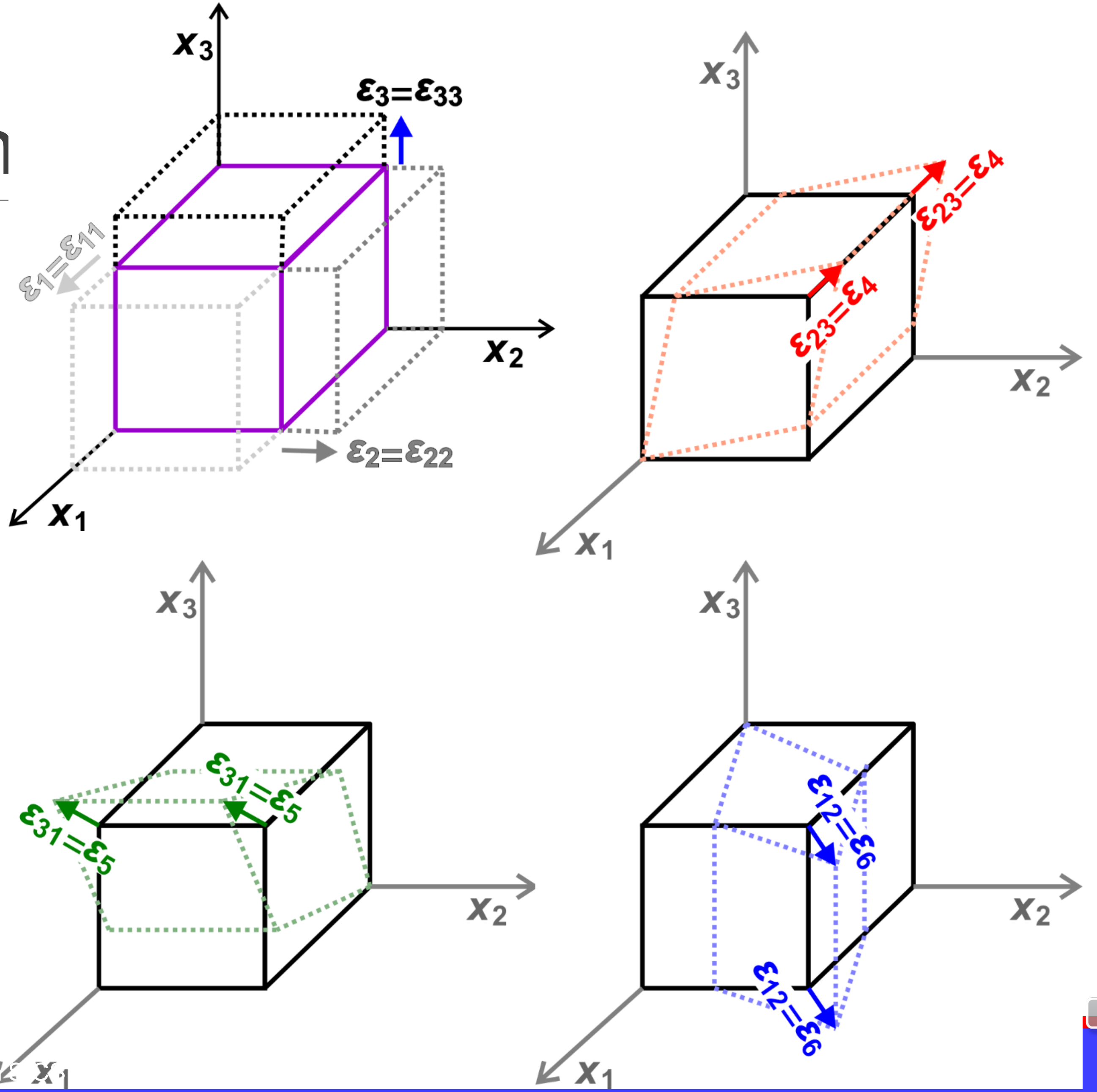


Voigt notation - strain

- ❑ 11 → 1, 22 → 2, 33 → 3, 23, 32 → 4, 13, 31 → 5, 12, 21 → 6
- ❑ Strain

➢ $\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \rightarrow$

$$\begin{bmatrix} \varepsilon_{1} & \varepsilon_{6} & \varepsilon_{5} \\ \varepsilon_{6} & \varepsilon_{2} & \varepsilon_{4} \\ \varepsilon_{5} & \varepsilon_{4} & \varepsilon_{3} \end{bmatrix} \quad (2.4.19)$$



Generalized Hooke's law by Voigt notation

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix} \quad (2.4.20)$$

✓ Why do the factors "2" appear to ε_4, ε_5 and ε_6?

The origin of the factors “2”

- Let us consider only one shear strain is formed.
- Original notation

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{32}\\\sigma_{31}\\\sigma_{13}\\\sigma_{12}\\\sigma_{21}\end{bmatrix}=\begin{bmatrix}C_{1111}&C_{1122}&C_{1133}&C_{1123}&C_{1132}&C_{1131}&C_{1113}&C_{1112}&C_{1121}\\C_{2211}&C_{2222}&C_{2233}&C_{2223}&C_{2232}&C_{2231}&C_{2213}&C_{2212}&C_{2221}\\C_{3311}&C_{3322}&C_{3333}&C_{3323}&C_{3332}&C_{3331}&C_{3313}&C_{3312}&C_{3321}\\C_{2311}&C_{2322}&C_{2333}&C_{2323}&C_{2332}&C_{2331}&C_{2313}&C_{2312}&C_{2321}\\C_{3211}&C_{3222}&C_{3233}&C_{3223}&C_{3232}&C_{3231}&C_{3213}&C_{3212}&C_{3221}\\C_{3111}&C_{3122}&C_{3133}&C_{3123}&C_{3132}&C_{3131}&C_{3113}&C_{3112}&C_{3121}\\C_{1311}&C_{1322}&C_{1333}&C_{1323}&C_{1332}&C_{1331}&C_{1313}&C_{1312}&C_{1321}\\C_{1211}&C_{1222}&C_{1233}&C_{1223}&C_{1232}&C_{1231}&C_{1213}&C_{1212}&C_{1221}\\C_{2111}&C_{2122}&C_{2133}&C_{2123}&C_{2132}&C_{2131}&C_{2113}&C_{2112}&C_{2121}\end{bmatrix}\begin{bmatrix}0\\0\\0\\\varepsilon_{23}\\\varepsilon_{32}\\0\\0\\0\\0\end{bmatrix} \quad (2.4.21)$$

- $\sigma_{ij}=C_{ij23}\varepsilon_{23}+C_{ij32}\varepsilon_{32}=C_{k4}\varepsilon_4+C_{k4}\varepsilon_4=2C_{k4}\varepsilon_4$ (2.4.22)

✓counted twice

The origin of the factors "2"

- Voigt notation with a instead of 2.
  - $$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\C_{21}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\C_{31}&C_{32}&C_{33}&C_{34}&C_{35}&C_{36}\\C_{41}&C_{42}&C_{43}&C_{44}&C_{45}&C_{46}\\C_{51}&C_{52}&C_{53}&C_{54}&C_{55}&C_{56}\\C_{61}&C_{62}&C_{63}&C_{64}&C_{65}&C_{66}\end{bmatrix}\begin{bmatrix}0\\0\\0\\a\,\varepsilon_4\\0\\0\end{bmatrix} \quad (2.4.23)$$
  - $$\sigma_i = aC_{i4}\varepsilon_4 \quad (2.4.24)$$
    - ✓ counted once

$$\begin{bmatrix}\varepsilon_{11}&\varepsilon_{12}&\varepsilon_{13}\\\varepsilon_{21}&\varepsilon_{22}&\varepsilon_{23}\\\varepsilon_{31}&\varepsilon_{32}&\varepsilon_{33}\end{bmatrix}$$

- In order to have equivalent elastic tensors, $a = 2$

Mineral Physics I
Chapter 2. Elasticity
Section 4. Generalized Hooke's law

End