Mineral Physics I
Chapter 2. Elasticity
Section 2. Strain

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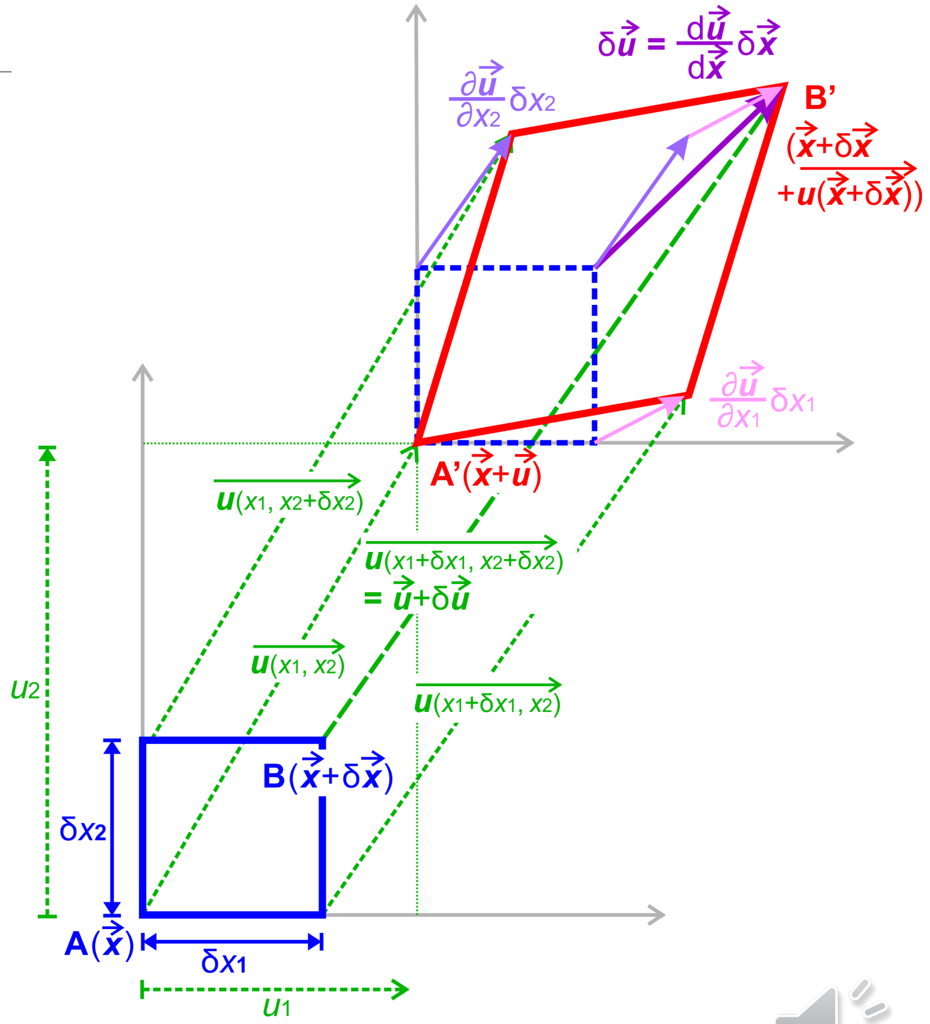
Terms

- **Displacement**: the difference between the final and initial positions of a point trajectory
- **Strain**: a description of deformation in terms of relative displacement of a part in the body that excludes rigid body-motions.

Displacement -1

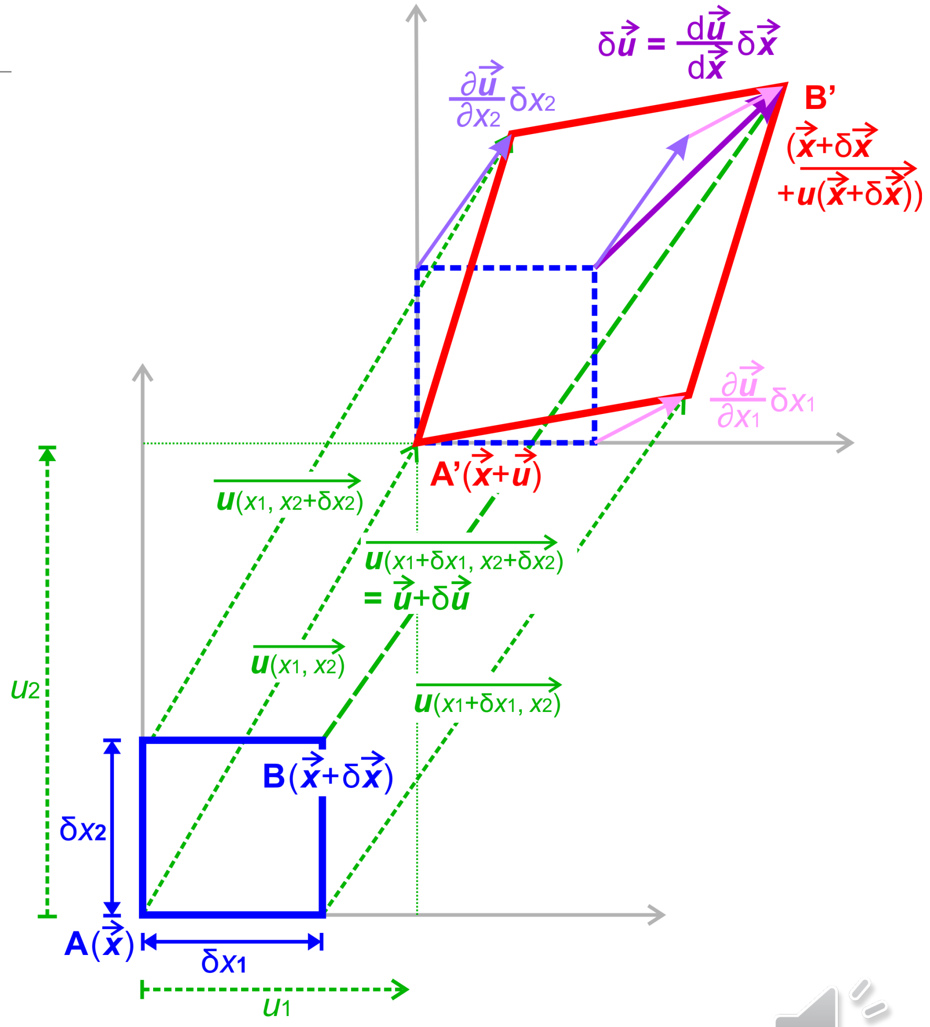
- ❑ Point A, before deformation,
  - ➢ $\vec{x} = (x_1, x_2, x_3)$ (2.2.1)
- ❑ Point B, adjacent to Point (A), before deformation
  - ➢ $\vec{x} + \delta\vec{x} = (x_1 + \delta x_1, x_2 + \delta x_2, x_3 + \delta x_3)$ (2.2.2)
- ❑ Point A is displaced to Point A' by deformation
  - ➢ $\overrightarrow{x'} = (x_1', x_2', x_3')$ (2.2.3)
- ❑ Point B is also displaced Point B' by deformation
  - ➢ $\overrightarrow{x'} + \delta\overrightarrow{x'} = (x_1' + \delta x_1',\ \ x_2' + \delta x_2',\ \ x_3' + \delta x_3')$ (2.2.4)

✓ $\delta\vec{x} \neq \delta\overrightarrow{x'}$



Displacement -2

- Displacement of Point A,
  - $\vec{u}(\vec{x}) = \overrightarrow{x'} - \vec{x}$ (2.2.5)
  - $\vec{u}(\vec{x}) = \left(u_1(\vec{x}), u_2(\vec{x}), u_3(\vec{x})\right)$
- Displacement at B,
  - $\vec{u}(\vec{x} + \delta\vec{x}) = \left(\overrightarrow{x'} + \delta\overrightarrow{x'}\right) - (\vec{x} + \delta\vec{x})$
  - $\vec{u}(\vec{x} + \delta\vec{x}) = \left(u_1(\vec{x} + \delta\vec{x}), u_2(\vec{x} + \delta\vec{x}), u_3(\vec{x} + \delta\vec{x})\right)$ (2.2.6)
- Difference of displacement between A and B
  - $\delta\vec{u} = \vec{u}(\vec{x} + \delta\vec{x}) - \vec{u}(\vec{x}) = (\mathrm{d}\vec{u}/\mathrm{d}\vec{x})\delta\vec{x}$ (2.2.7)

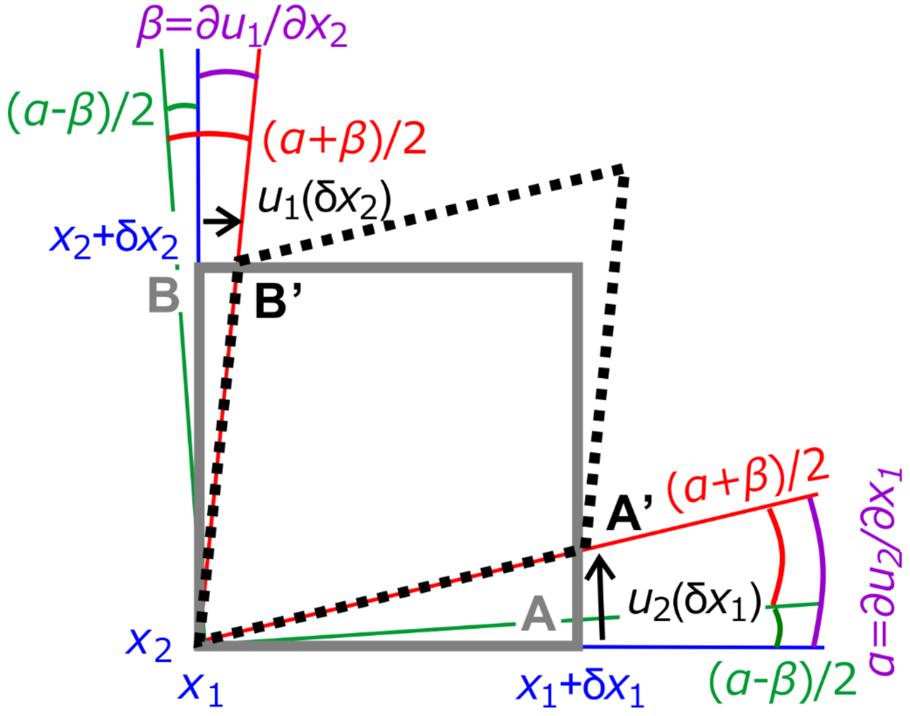


Strain tensor

- ❑ Consider infinitesimal deformation
  - ➢ Change in angle by A → A': α
    - ✓ $\sin\alpha \approx \alpha$

$$\alpha = \lim_{\delta x_1 \to 0} \frac{u_2(x_1+\delta x_1, x_2) - u_2(x_1, x_2)}{\delta x_1} = \frac{\partial u_2}{\partial x_1} \qquad (2.2.8)$$

  - ➢ Change in angle by B → B': $\beta = \frac{\partial u_1}{\partial x_2}$
  - ➢ Angle $(\alpha - \beta)/2$: rotation, no deformation
  - ➢ Angle $(\alpha + \beta)/2$: deformation
    - ✓ We should consider this amount



- ❑ 2nd-rank tensor of infinitesimal strain, ε_{ij} $(i, j = 1, 2, 3)$
  - ➢ Strain tensor: $\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (2.2.9)
    - ✓ $\varepsilon_{ij} = \varepsilon_{ji}$: symmetric tensor (2.2.10)

Strain tensor

- Consider infinitesimal deformation
  - Change in angle by A → A': α
    - ✓ $\sin\alpha \approx \alpha$

$$\alpha = \lim_{\delta x_1 \to 0} \frac{u_2(x_1+\delta x_1, x_2) - u_2(x_1, x_2)}{\delta x_1} = \frac{\partial u_2}{\partial x_1} \qquad (2.2.8)$$

  - Change in angle by B → B': $\beta = \frac{\partial u_1}{\partial x_2}$
  - Angle $(\alpha-\beta)/2$: rotation, no deformation
  - Angle $(\alpha+\beta)/2$: deformation
    - ✓ We should consider this amount
- 2nd-rank tensor of infinitesimal strain, ε_{ij} $(i, j = 1, 2, 3)$
  - Strain tensor: $\varepsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ (2.2.9)
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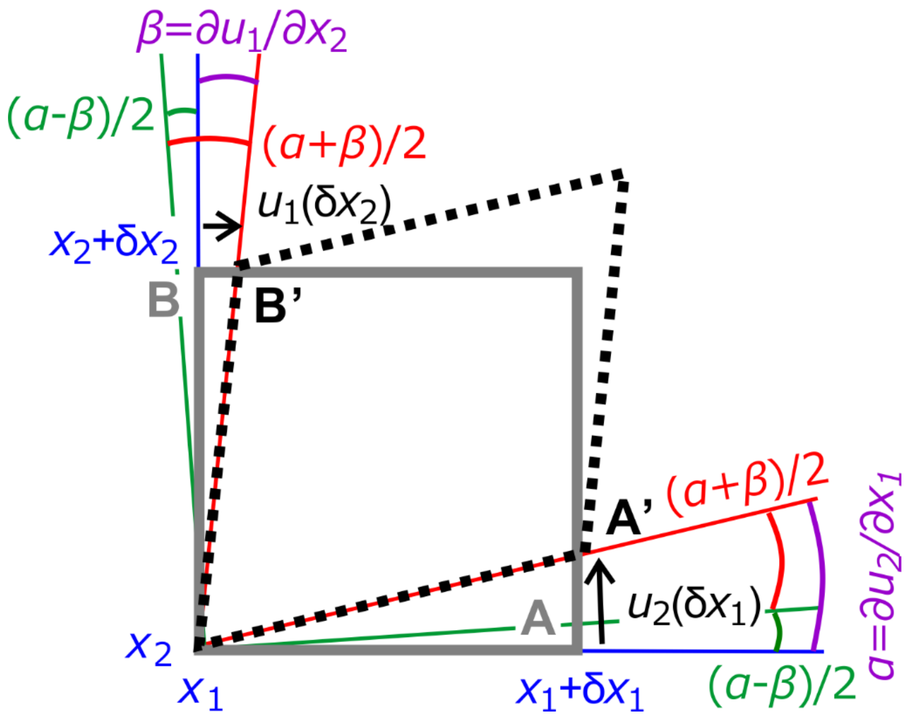


Strain Matrix

- $[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right) \end{bmatrix}$ (2.2.11)

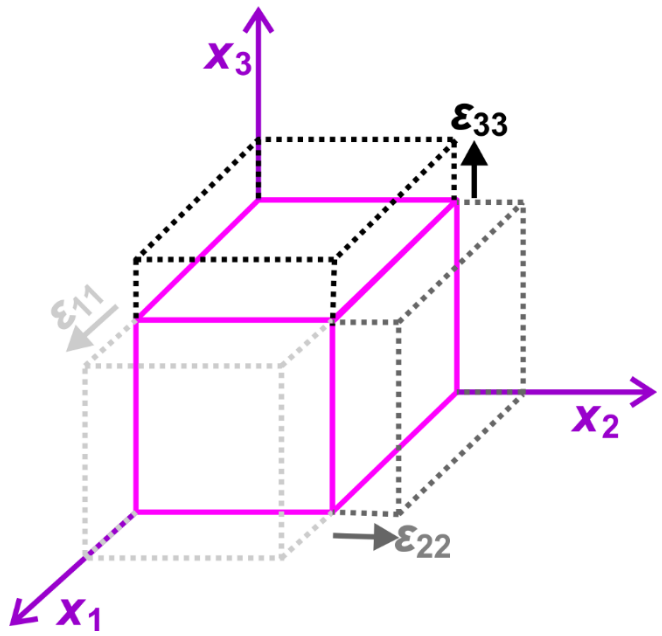
$$= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \quad (2.2.11')$$

- The same color : the same values

Normal strains

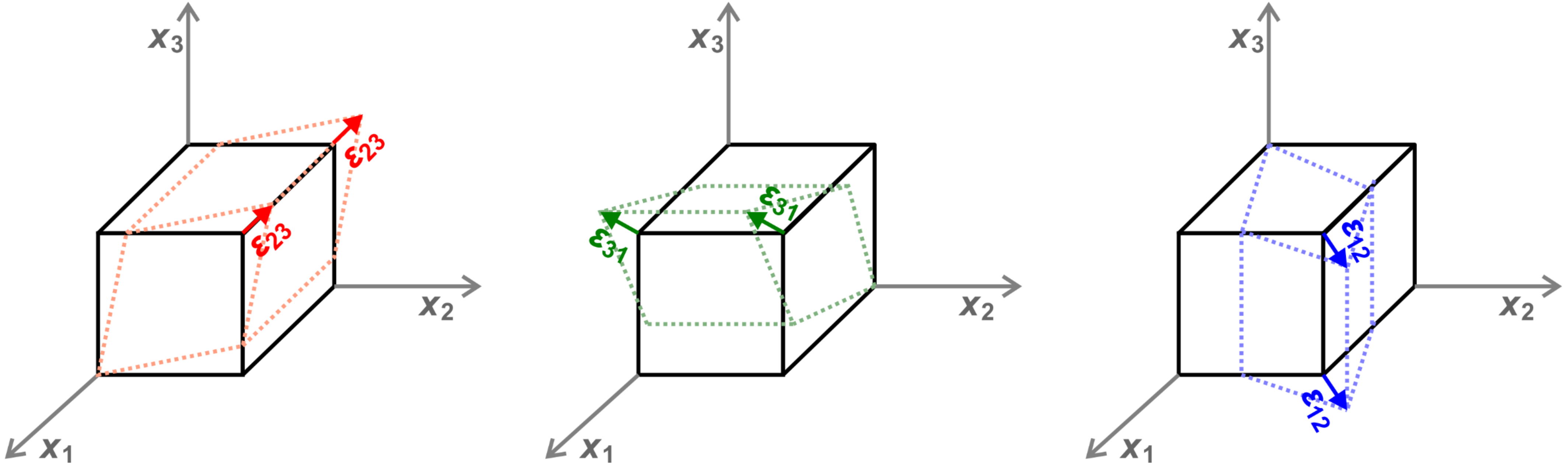
- $[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} =$

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$



- The diagonal components ε_{11}, ε_{22}, ε_{33}: expansion in the x_1, x_2, x_3, directions.
  - Normal strain or compressional strain
    - ✓ Although the plus sign is expansion.

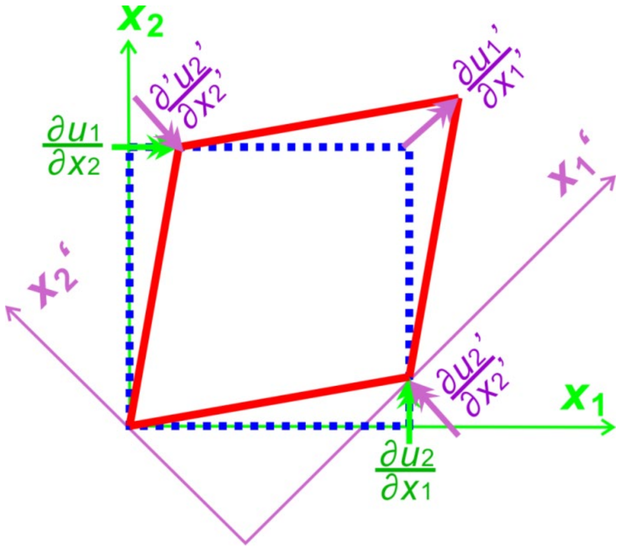
Shear strains



- $\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$, $\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$, $\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$

Orthogonalization

- ❑ Strain tensor can be diagonalized by a coordinate transformation
  - ➢ Characteristic value problem of the symmetric matrix
  - ➢ $[A]^{-1}[\varepsilon_{ij}][A] = \begin{bmatrix} \frac{\partial u_1'}{\partial x_1'} & 0 & 0 \\ 0 & \frac{\partial u_2'}{\partial x_2'} & 0 \\ 0 & 0 & \frac{\partial u_3'}{\partial x_3'} \end{bmatrix}$ (2.2.12)
  - ➢ Strain tensor; symmetric → the characteristic vectors (directions of the expansion and contraction) are orthogonal
  - ➢ Any strain is equivalent to a combination of expansion and contraction

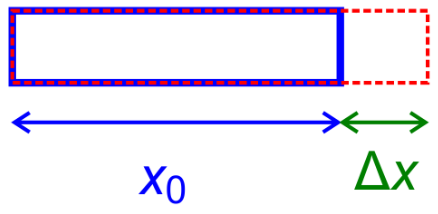


- ❑ Dilatation: relative volume change ($\Delta V/V$)
  - ➢ $\frac{\Delta V}{V} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ (2.2.13)

Primary definitions of normal and shear strains

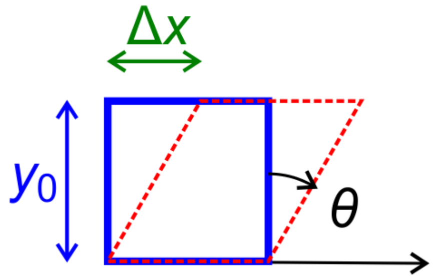
- Normal or compressional strain: ε
  - $\varepsilon \equiv \Delta x / x_0$ (2.2.14)
  - ε is positive when the object elongated
    - ✓ Opposite sense to the word “compressional”
    - ✓ However, it is often obscure
    - ✓ As shown later, it is apparently opposite to the direction of stress

Compressional strain



- Shear strain: γ
  - $\gamma \equiv \Delta x / y_0$ (2.2.15)
  - $\theta = \mathrm{atan}(\gamma)$ (2.2.16)

Shear strain



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