Mineral Physics I
Chapter 2. Elasticity

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This chapter

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Mineral Physics I
Chapter 2. Elasticity
Section 1. Mathematical and physical backgrounds

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- Characteristic values and vectors of linear transformation
- Complex number and function
- Simple differential equation
- 1D Wave function
- Vector operators
- Plane wave in 3D space

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- **Characteristic values and vectors of linear transformation**
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Characteristic values and vectors -1

- Some linear transformation $[A]$ can be decomposed into expansion and contraction
  - The directions of expansion/contraction,
    - ✓ $\vec{x}$: characteristic vector (eigenvector)
  - The magnitude of expansion/contraction,
    - ✓ λ: characteristic values (eigenvalue)
  - $[A]\vec{x} = \lambda\vec{x}$ (2.1.1)

- A linear transformation in an n-D space has n pairs of characteristic vectors and values

$$\begin{gathered} [A]\overrightarrow{x_1} = \lambda_1 \overrightarrow{x_1} \\ \vdots \\ [A]\overrightarrow{x_n} = \lambda_n \overrightarrow{x_n} \end{gathered} \quad (2.1.2)$$

Characteristic values and vectors -2

- Eq (2.1.2) can be expressed using $[\lambda] = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, $[X] = [\overrightarrow{x_1} \quad \ldots \quad \overrightarrow{x_n}]$,
  - $[A][X] = [X][\lambda]$
  - $[X]^{-1}[A][X] = [X]^{-1}[X][\lambda] = [\lambda]$ (2.1.3)
    - ✓ diagonalization

- Example: $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$
  - $\lambda = 2, 3,\ \overrightarrow{x_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \overrightarrow{x_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ [\lambda] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},\ [X] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ [X]^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$
  - $[X]^{-1}[A][X] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = [\lambda]$

Diagonalization of symmetric matrix

- Transposed matrix, A^T: a matrix whose components are flipped from the original matrix, A over the diagonal. *ex.* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.
  - $[A^T]_{ij} = [A]_{ji}$ (2.1.4)
  - Also for vectors: transformation from the horizontal vector to the vertical vector, $(a \quad b)^T = \begin{pmatrix} a \\ b \end{pmatrix}$, and vice versa.
- Symmetric matrix: *ex.* $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$
  - $A = A^T$ or $a_{ij} = a_{ji}$ (2.1.5)
- An important theorem: the characteristic vectors of a symmetric matrix are orthogonal to each other.
  - $\overrightarrow{x_i} \perp \overrightarrow{x_j} \qquad (i \neq j)$ (2.1.6)

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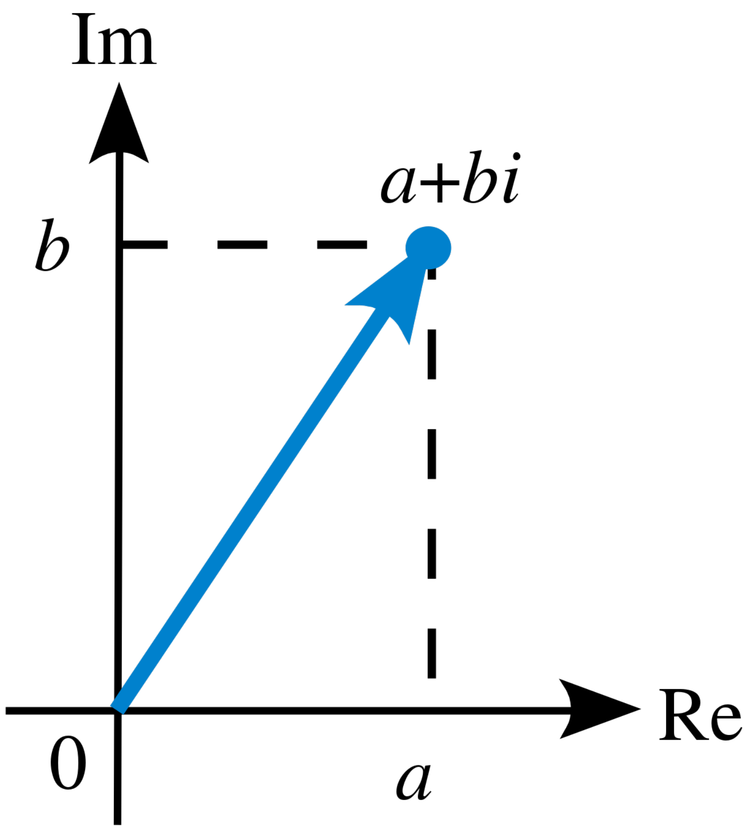
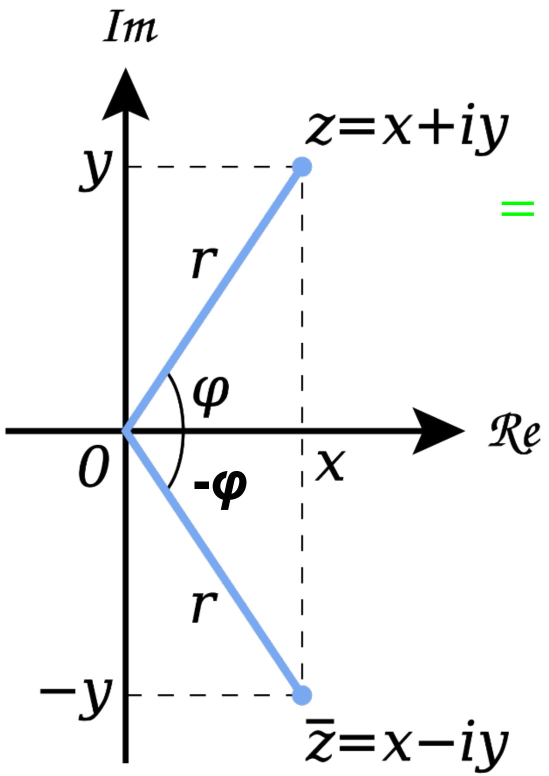
Complex number and function -1

- **Complex number**
  - $z \equiv a + bi = \mathrm{Re}(z) + \mathrm{Im}(z)\mathrm{i}$ (2.1.7)
    - ✓ $\mathrm{i} \equiv \sqrt{-1}$: imaginary unit,
    - ✓ a: real part, $\mathrm{Re}(z)$,
    - ✓ b: imaginary part, $\mathrm{Im}(z)$
- **Argand diagram**
  - Horizontal axis: $\mathrm{Re}(z)$
  - Vertical axis: $\mathrm{Im}(z)$
  - A complex number $z = a + bi$ → point (a, b)



Complex number and function -2

- Complex conjugate
  - ✓ $z = x + \mathrm{i}y \rightarrow z^* = x - \mathrm{i}y$ (2.1.8)
- Modulus (or absolute value or magnitude)
  - ➢ $r = |z| \equiv \sqrt{x^2 + y^2}$ (2.1.9)
    - ✓ The distance from the original point.
- Polar notation
  - ➢ $z = r(\cos\varphi + \mathrm{i}\sin\varphi)$ (2.1.10)
    - ✓ $\cos\varphi = x/r, \ \sin\varphi = y/r$
    - ✓ φ: Argument, the angle of the OZ with the real axis.
  - ➢ $z^* = r(\cos\varphi - \mathrm{i}\sin\varphi) = r(\cos(-\varphi) + \mathrm{i}\sin(-\varphi))$ (2.1.11)



$= r(\cos\varphi + \mathrm{i}\sin\varphi)$

$= r(\cos\varphi - \mathrm{i}\sin\varphi)$

Complex number and function -3

- Complex exponential function
  - $\exp(\pm i\theta) = e^{\pm i\theta} \equiv \cos\theta \pm i\sin\theta$ (2.1.12)
  - $e^{\phi\pm i\theta} = e^{\phi}(\cos\theta \pm i\sin\theta)$ (2.1.13)
  - $\left(e^{i\theta}\right)^{*} = \cos\theta - i\sin\theta = e^{-i\theta}$ (2.1.14)
  - $\left|e^{i\theta}\right| \equiv \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1$ (2.1.15)
- The complex exponential function satisfies the basic property of exponential functions, $e^{a}e^{b} = e^{a+b}$
  - $e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

    $= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$

    $= \cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)}$ (2.1.16)

Complex number and function -4

- Relations between the complex exponential and real trigonometric functions
  - $\exp(\mathrm{i}x) = \cos x + \mathrm{i} \sin x$ (2.1.12)
  - $\exp(-\mathrm{i}x) = \cos x - \mathrm{i} \sin x$ (2.1.12')
  - [(2.1.12)+(2.1.12')]/2: $\cos x = [\exp(ix) + \exp(-ix)]/2$ (2.1.17)
  - [(2.1.12)-(2.1.12')]/2: $\sin x = [\exp(ix) - \exp(-ix)]/2$ (2.1.18)

Complex number and function -5

- The complex exponential function $\exp\left(\mathrm{i}\frac{2\pi}{T}x\right) = \cos\left(\frac{2\pi}{T}x\right) + \mathrm{i}\sin\left(\frac{2\pi}{T}x\right)$ is more convenient to express an oscillation than the real trigonometric function
  - Containing the trigonometric functions
  - A periodic function of a period of T
  - Calculation is easier ($e^{\mathrm{i}\alpha}e^{\mathrm{i}\beta} = e^{\mathrm{i}(\alpha+\beta)}$)

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Simple differential equation

- The differential equation that we will encounter in this chapter:
  - $\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} = -A^2\psi$: (2.1.19)
    - ✓ The equation of the harmonic oscillators.
  - The solution of (2.1.19)
  - $\psi = C\mathrm{e}^{\pm iAx}$ (2.1.20)
    - ✓ $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(Ce^{\pm iAx}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(Ce^{\pm iAx}\right)$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left\{(\pm iA)Ce^{\pm iAx}\right\} = (\pm iA)^2Ce^{\pm iAx}$$

$$= -A^2Ce^{\pm iAx} = -A^2\psi$$

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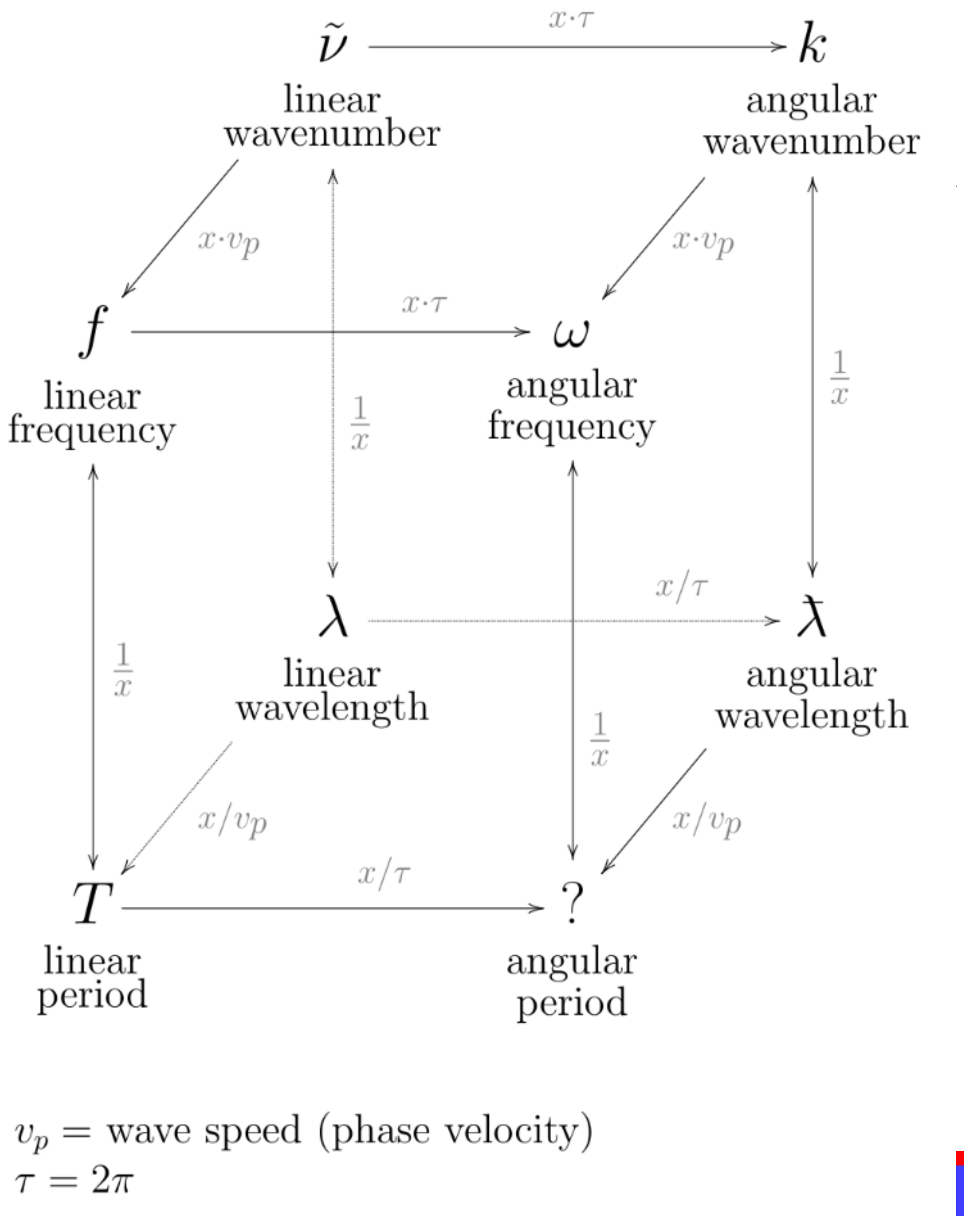
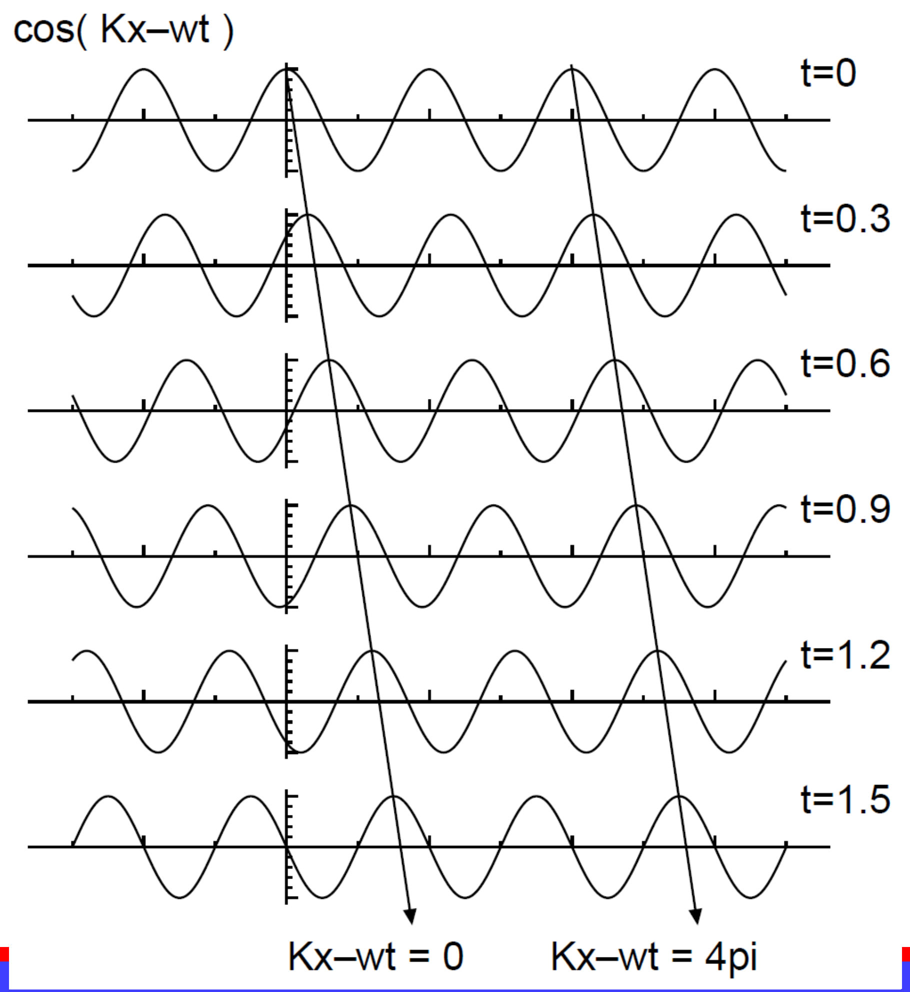
Wave function -1

- Valuables for wave
  - v: phase velocity
  - T: period
  - $f = \frac{1}{T}$: frequency
  - $\omega = 2\pi f$: angular frequency
  - λ: wave length
  - $\tilde{\nu} = 1/\lambda$: wave number
  - $k = 2\pi\tilde{\nu}$: angular wave number
  - A: amplitude



Wave function -2

- ❑ Wave: a medium oscillates periodically at each point, and that the peaks and nodes shift with time at a constant rate
  - ➢ The medium oscillates both spatially and timely



Wave function -3

- Wave function
  - $Y(x,t) = A \exp\left(\mathrm{i}\frac{2\pi}{\lambda}x\right) \exp(\pm\mathrm{i}\frac{2\pi}{T}t)$ (2.1.21)

    $= A \exp(\mathrm{i}2\pi\tilde{\nu}x) \exp(\pm\mathrm{i}2\pi f t)$ (2.1.21')

    $= A \exp(\mathrm{i}kx) \exp(\pm\mathrm{i}\omega t)$ (2.1.21'')

    $= A \exp[\mathrm{i}(kx \pm \omega t)]$ (2.1.22)

  If $kx - \omega t$, the waves propagate in the positive direction

  If $kx + \omega t$, the waves propagate in the negative direction

- Phase velocity, v
  - $v = f\lambda = \omega/k$ (2.1.23)
  - $Y(x,t) = A \exp[\mathrm{i}k(x \pm vt)]$ (2.1.24)

Wave function -4

- **Wave equation**
  - $\frac{\partial^2 Y}{\partial t^2} = v^2 \frac{\partial^2 Y}{\partial x^2}$ (2.1.25)
  - Substituting (2.1.21) $Y(x,t) = A\exp(ikx)\exp(\pm i\omega t)$ into (2.1.25)
    - ✓ Left side: $\frac{\partial^2 Y}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} A\exp(ikx)\exp(\pm i\omega t)\right]$

      $= \frac{\partial}{\partial t}\left[-i\omega A\exp(ikx)\exp(\pm i\omega t)\right] = -\omega^2 A\exp(ikx)\exp(\pm i\omega t) = -\omega^2 Y$
    - ✓ Right side: $v^2 \frac{\partial^2 Y}{\partial x^2} = v^2(-k^2 A\exp(ikx)\exp(\pm i\omega t)) = -\left(\frac{\omega}{k}\right)^2 k^2 Y = -\omega^2 Y$
    - ✓ Left = Right: (2.1.21) is a solution of (2.1.25)

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Vector operators -1

- Vector $\vec{a} = (a_1 \quad a_2 \quad a_3)$
  - a_1, a_2 and a_3 are functions of position coordinates $(x_1 \quad x_2 \quad x_3)$
- Scalar b
  - A function of position coordinates $(x_1 \quad x_2 \quad x_3)$
- Operator "nabla"
  - $$\nabla \equiv \left(\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3}\right) \qquad (2.1.26)$$
- Gradient
  - $$\nabla b \equiv \left(\frac{\partial b}{\partial x_1} \quad \frac{\partial b}{\partial x_2} \quad \frac{\partial b}{\partial x_3}\right) \qquad (2.1.27)$$
    - ✓ b: scalar, ∇b: vector

Vector operators -2

- Divergence
  - $\mathrm{div}(\vec{a}) = \nabla \cdot \vec{a} \equiv \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$ (2.1.28)
    - ✓ $\vec{a}$: vector, $\mathrm{div}(\vec{a})$: scalar
- Rotation
  - $\mathrm{rot}(\vec{a}) = \mathrm{curl}(\vec{a}) = \nabla \times \vec{a} \equiv \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \quad \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \quad \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_3} \right)$ (2.1.29)
    - ✓ $\vec{a}$: vector, $\mathrm{rot}(\vec{a})$: vector
- Laplacian
  - $\nabla^2 b = \nabla \cdot \nabla b \equiv \frac{\partial^2 b}{\partial x_1^2} + \frac{\partial^2 b}{\partial x_2^2} + \frac{\partial^2 b}{\partial x_3^2}$ (2.1.30)
    - ✓ b: scalar, $\nabla^2 b$: scalar

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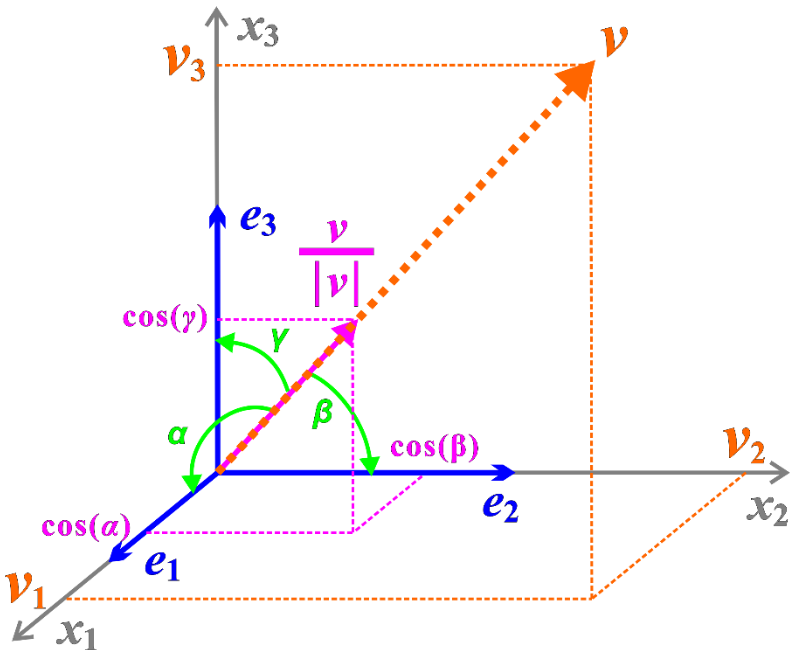
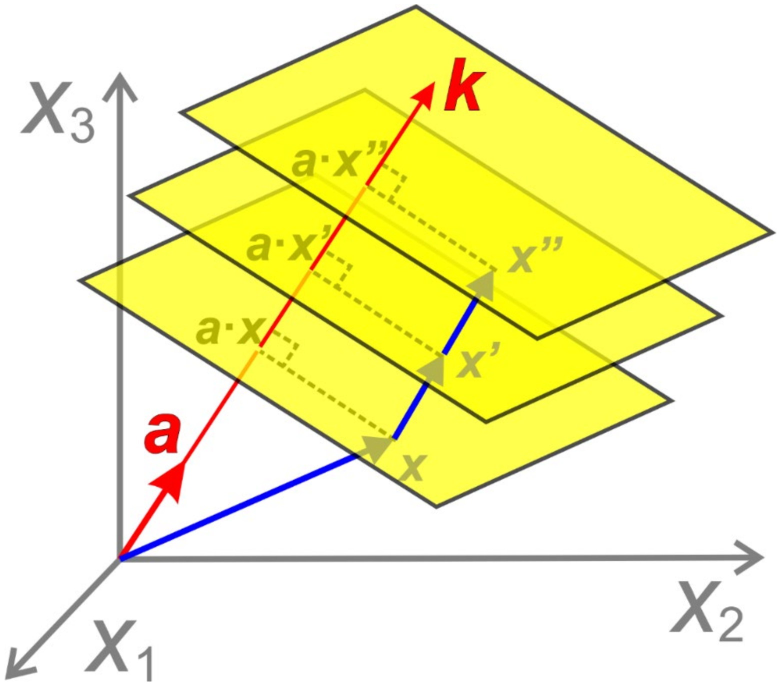
Directional cosine

- $\boldsymbol{v}$: a vector in 3D Euclidean space
  - $\boldsymbol{v} = (v_1 \quad v_2 \quad v_3) = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$ (2.1.31)
    - ✓ $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$: the standard basis vector in Cartesian notation
- The angles that $\boldsymbol{v}$ forms with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$: α, β, γ
  - $\cos\alpha = \frac{\boldsymbol{e}_1 \cdot \boldsymbol{v}}{|\boldsymbol{v}|} = \frac{v_1}{\sqrt{v_1^2+v_2^2+v_3^2}}$ (2.1.32a)
  - $\cos\beta = \frac{\boldsymbol{e}_2 \cdot \boldsymbol{v}}{|\boldsymbol{v}|} = \frac{v_2}{\sqrt{v_1^2+v_2^2+v_3^2}}$ (2.1.32b)
  - $\cos\gamma = \frac{\boldsymbol{e}_3 \cdot \boldsymbol{v}}{|\boldsymbol{v}|} = \frac{v_3}{\sqrt{v_1^2+v_2^2+v_3^2}}$ (2.1.32c)
  - $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{a} = |\boldsymbol{v}|(\cos\alpha \quad \cos\beta \quad \cos\gamma)$ (2.1.33)
    - ✓ $\boldsymbol{a} = (\cos\alpha \quad \cos\beta \quad \cos\gamma)$: **directional cosine**
  - $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{v_1^2+v_2^2+v_3^2}{v_1^2+v_2^2+v_3^2} = 1$ (2.1.34)



Plane wave function in 3D -1

- Plane wave propagating in the direction $\boldsymbol{a} = (a_1 \quad a_2 \quad a_3) = (\cos\alpha \quad \cos\beta \quad \cos\gamma)$.
  - ✓ Directional cosine
  - ➢ The equation of the points of the wave surface $\boldsymbol{x} = (x_1 \quad x_2 \quad x_3)$:
    - ✓ $\boldsymbol{a} \cdot \boldsymbol{x} = const.$ (2.1.35)
      - The $\boldsymbol{e}$-direction component of the position vector $\boldsymbol{r}$ is identical anywhere on the wave surface
    - ✓ $a_1x_1 + a_2x_2 + a_3x_3 = const.$ (2.1.36)
  - ➢ A 3D variable (displacement in elasticity) $\boldsymbol{Y}$ on the wave surface propagating with the speed v is identical.
    - ✓ $\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{a} \cdot \boldsymbol{x} - vt)$ (2.1.37)

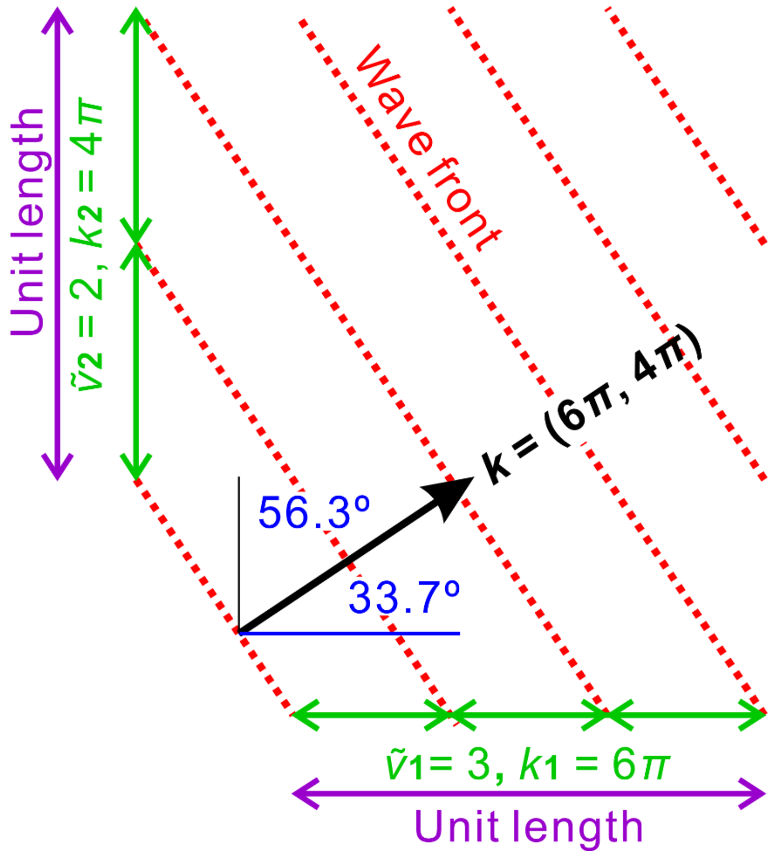


Plane wave function in 3D -2

- If the wave is sinusoidal, the function of $\boldsymbol{Y}$,
  - Eq. (2.1.37) $\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{a} \cdot \boldsymbol{x} - vt) \rightarrow \boldsymbol{Y} = \boldsymbol{A} \exp[\mathrm{i}k(\boldsymbol{a} \cdot \boldsymbol{x} - vt)]$ (2.1.38)
    - ✓ $\because$ (2.1.24): $Y(x, t) = A \exp[\mathrm{i}k(x - vt)]$
    - ✓ $\boldsymbol{Y} = \boldsymbol{A} \exp[\mathrm{i}k(a_1 x_1 + a_2 x_2 + a_3 x_3 - vt)]$
- If the wave propagates in the x_i direction, $a_{j \neq i} = 0$. Eq. (2.1.38) becomes
  - $\boldsymbol{Y} = \boldsymbol{A} \exp[\mathrm{i}(k a_i x_i - kvt)] = \boldsymbol{A} \exp[\mathrm{i}(k_i x_i - \omega t)]$
    - ✓ $k_i = k a_i = k \cos \alpha$: the wave number in the i direction
- $\boldsymbol{k} = (k_1 \quad k_2 \quad k_3) = (k a_1 \quad k a_2 \quad k a_3)$: wave vector
  - $\boldsymbol{Y} = \boldsymbol{A} \exp[\mathrm{i}(k_1 x_1 + k_2 x_2 + k_3 x_3 - \omega t)] = \boldsymbol{A} \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]$ (2.1.39)

An example of a 2D plane wave

- 2D example
  - $\boldsymbol{k} = (6\pi \quad 4\pi) = 2\pi\sqrt{13}\left(\frac{3}{\sqrt{13}} \quad \frac{2}{\sqrt{13}}\right) = 2\pi\sqrt{13}(\cos 33.7° \quad \cos 56.3°)$
    - ✓ $\tilde{\nu}_1 = 3$: 3 waves in a unit length in the x_1 direction
    - ✓ $\tilde{\nu}_2 = 2$: 2 waves in a unit length in the x_2 direction
  - The wave propagates in the direction with a larger wave number



3D wave equation -1

- Find a partial differential equation that a function $Y = f(a_1x_1 + a_2x_2 + a_3x_3 - vt)$ satisfies.
  - Let $\chi = a_1x_1 + a_2x_2 + a_3x_3 - vt \rightarrow Y = f(\chi)$
  - $\frac{\partial^2 Y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\mathrm{d}f}{\mathrm{d}\chi}\frac{\partial \chi}{\partial t}\right) = \frac{\partial}{\partial t}\left(-v\frac{\mathrm{d}f}{\mathrm{d}\chi}\right) = -v\frac{\partial}{\partial t}\frac{\mathrm{d}f}{\mathrm{d}\chi} = v^2\frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2}$ (2.1.40)
  - $\frac{\partial^2 Y}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\mathrm{d}f}{\mathrm{d}\chi}\frac{\partial \chi}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(a_1\frac{\mathrm{d}f}{\mathrm{d}\chi}\right) = a_1\frac{\partial \chi}{\partial x_1}\frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2} = a_1^2\frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2}$ (2.1.41a)

  Similarly,
  - $\frac{\partial^2 Y}{\partial x_2^2} = a_2^2\frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2}$ (2.1.41b)
  - $\frac{\partial^2 Y}{\partial x_3^2} = a_3^2\frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2}$ (2.1.41c)
    - ✓ $\because \frac{\partial \chi}{\partial t} = -v,\ \frac{\partial \chi}{\partial x_1} = a_1,\ \frac{\partial \chi}{\partial x_2} = a_2,\ \frac{\partial \chi}{\partial x_3} = a_3$

3D wave equation -2

- The sum of (2.1.41a), (2.1.41b), and (2.1.41c)
  - $\frac{\partial^2 Y}{\partial x_1^2} + \frac{\partial^2 Y}{\partial x_2^2} + \frac{\partial^2 Y}{\partial x_3^2} = a_1^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2} + a_2^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2} + a_3^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2}$

    $= (a_1^2 + a_2^2 + a_3^2) \frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2}$ (2.1.42)
    - ✓ $a_1^2 + a_2^2 + a_3^2 = 1$
      - $\because \boldsymbol{a} = (a_1 \quad a_2 \quad a_3) = (\cos\alpha \quad \cos\beta \quad \cos\gamma)$: the directional cosine
  - $v^2 \left(\frac{\partial^2 Y}{\partial x_1^2} + \frac{\partial^2 Y}{\partial x_2^2} + \frac{\partial^2 Y}{\partial x_3^2} \right) = v^2 \frac{\mathrm{d}^2 f}{\mathrm{d}\chi^2} = \frac{\partial^2 Y}{\partial t^2}$ (2.1.43)
  - $\frac{\partial^2 Y}{\partial t^2} = v^2 \left(\frac{\partial^2 Y}{\partial x_1^2} + \frac{\partial^2 Y}{\partial x_2^2} + \frac{\partial^2 Y}{\partial x_3^2} \right) = v^2 \nabla^2 Y$ (2.1.44)
    - ✓ 3D wave equation

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