3. Lattice vibration

5. Phase and group velocities

5.1 Group velocity and phase velocity

Group velocity (v_g) is the velocity of the propagation of the periodic amplitude variation, and phase velocity (v_p) is the velocity of the phase propagation of each wave. The periodic variation in amplitude, beat, results from interference between two waves of slightly different frequencies. Considering two waves, $f_1(x,t)$ and $f_2(x,t)$, written as

$$f_1(x,t) = \exp[i(k_1 x - \omega_1 t)] \qquad (5.1.1a)$$

$$f_2(x,t) = \exp[i(k_2 x - \omega_2 t)], \qquad (5.1.1b)$$

where ω and k are angular frequency and angular wave number, respectively, the superposition $f(x,t)$ of these two waves is expressed as

$$\begin{aligned} f(x,t) &= f_1(x,t) + f_2(x,t) \\ &= \{\cos[(k_1 x - \omega_1 t)] + i\sin[(k_1 x - \omega_1 t)]\} \\ &\quad + \{\cos[(k_2 x - \omega_2 t)] + i\sin[(k_2 x - \omega_2 t)]\} \\ &= 2\cos\left[\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right]\cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right] \\ &\quad + 2i\sin\left[\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right]\cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right] \\ &= 2\cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right]\exp\left[i\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right]. \end{aligned} \qquad (5.1.2)$$

The factor of the cosine function and complex exponential function in Eq. (5.1.2) represent beat and phase, respectively. Figure 1 shows the superposition of two waves. The amplitude variation ($f_b(x,t)$) is

$$\begin{aligned} f_b(x,t) &= \cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right] \\ &= \cos\left[\frac{\Delta k \cdot x - \Delta\omega \cdot t}{2}\right] \\ &\cong \cos\left[\frac{\Delta k \cdot x - \frac{d\omega}{dk}\Delta k \cdot t}{2}\right] \\ &= \cos\left[\frac{\Delta k}{2}\left(x - \frac{d\omega}{dk}t\right)\right], \end{aligned} \qquad (5.1.3)$$

where Δk $(= k_1 - k_2)$ and $\Delta\omega$ $(= \omega_1 - \omega_2)$ are differential angular wave number and differential angular frequency, respectively. The velocity of this beat, group velocity (v_g), is

$$v_g = \frac{d\omega}{dk}. \qquad (5.1.4)$$

The group velocity can be positive, negative or zero. From Eq. (5.1.2) the oscillation ($f_p(x,t)$) within the beat is

$$\begin{aligned} f_p(x,t) &= \exp\left[i\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right] \\ &= \exp\left[i\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\right] \\ &= \exp\left[i\frac{k_1 + k_2}{2}\left(x - \frac{\omega_1 + \omega_2}{k_1 + k_2}t\right)\right]. \end{aligned} \qquad (5.1.5)$$

The velocity of this wave is called as phase velocity (v_p) and written as

$$v_p = \frac{\frac{\omega_1 + \omega_2}{2}}{\frac{k_1 + k_2}{2}} = \frac{\overline{\omega}}{\overline{k}}, \tag{5.1.6}$$

where $\overline{\omega}$ and $\bar{k}$ are average angular frequency and average angular wave number, respectively. Figure 1 shows the wave propagation of the periodic amplitude variation and of each wave.

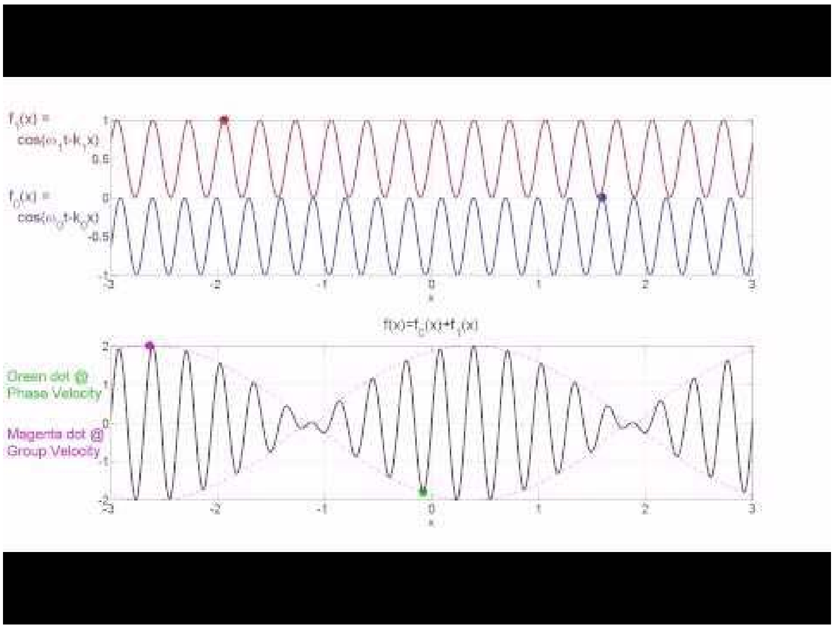

Fig. 1. A superposition of two waves. (Top) The first wave. (Middle) The second wave. (Bottom) A black curve represents a summation of the two waves, propagating with the phase velocity. Green curves represent envelopes of the blue curve, propagating with the group velocity.

5.2 Dispersion relation

A dispersion relation represents the relation between angular frequency (ω) and angular wave number (k). Thus the dispersion relation is written as

$$\omega = f(k). \tag{5.1.7}$$

Figure 2 shows an example of a dispersion curve. Given an angular wave number, the group velocity is obtained from the slope of this curve, and the phase velocity is obtained from a division of ω ($= f(k)$) and k.

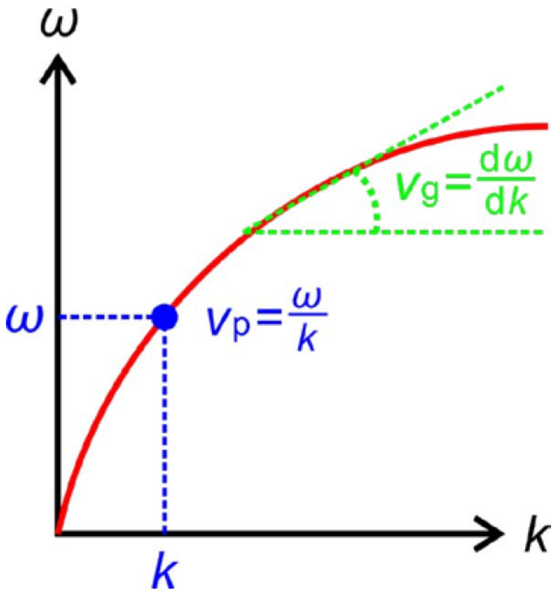

Fig. 2. An example of a dispersion curve. A red curve represents the dispersion relation $\omega = f(k)$. Green lines represent the slope of the curve at an angular wave number. This slope equals the group velocity v_g. Blue lines represent the angular frequency at an angular wave number. The phase velocity v_p equals the division of the angular frequency and the angular wave number.