2. Elasticity

3. Stress

3.1 Stress tensor

A stress is a physical quantity used to express the magnitude and/or direction of action of a force generated inside of an object. In static problems, the condition of stress is that the problem must be in a state of equilibrium. The stress is the force per unit area. The stress is applied across a small boundary. The boundary can be defined on the body's surface and anywhere within the body. Also, the stress is a measure of a function to deform the body. The stresses are related to two directions. One is a direction of the stress itself, and the other is the direction of the boundary. The stress component σ are expressed using two subscripts. If the stress component in the x_i direction and across the boundary normal to the x_j direction, it is written as σ_{ij}. This σ is referred to the stress tensor.

The stress tensor is the 2nd-rank tensor as well as the strain tensor. The stress is normal to the boundary of interest, if the two subscripts are identical, in other words $i = j$. This kind of stresses is referred to as the normal stresses. The normal stresses are positive when stress is generated outward on each surface of the object. In other words, positive stresses mean tensile stress and negative stresses mean compressive stress. The stress is parallel to the boundary of the interest, if the two subscripts are different, in other words $i \neq j$. This kind of stresses is referred to as the shear stresses. The shear stresses are positive in the positive direction of the coordinate axis x_j when the outward normal of the plane perpendicular to the coordinate axis x_i points in the positive direction.

Of the stress tensor, the stress direction acting perpendicular to the plane is a normal stress and the remaining component is a shear stress. If we focus on the plane normal to the x_1 axis direction, σ_{11} is in the x_1 direction and normal to this plane, therefore σ_{11} is the normal stress. σ_{21} and σ_{31} are parallel to this plane and in the x_2 axis direction and x_3 axis direction respectively, therefore these stresses are the shear stresses. The similar stress relationship is also found in the x_2 axis and x_3 axis directions. For example, if we focus on the plane normal to the x_2 axis direction, σ_{22} is in the x_2 axis direction and normal to this plane, therefore σ_{22} is the normal stress. σ_{12} and σ_{32} are parallel to this plane and in the x_1 axis direction and x_3 axis direction respectively, therefore these stresses are the shear stresses.

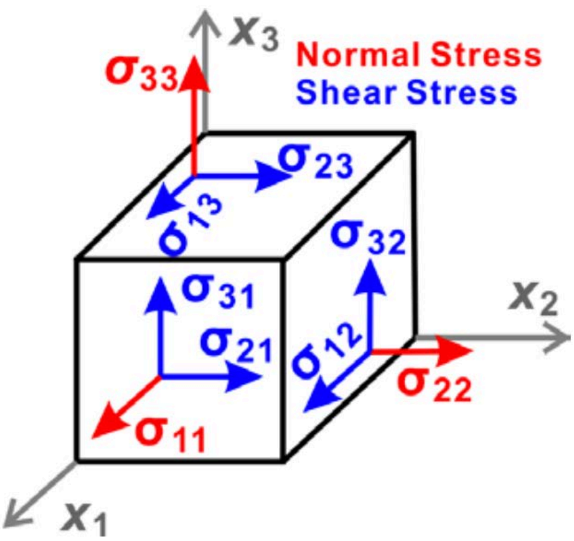

Fig. 1. The stresses to the cubic body. Three axes are the x_1 axis direction, the x_2 axis direction, and the x_3 axis direction, respectively. The red stress tensor (σ_{11}, σ_{22} and σ_{33}) indicate the normal stress for each plane. The blue stress tensor (σ_{21}, σ_{31}, σ_{12}, σ_{32}, σ_{13}, and σ_{23}) indicate the shear stress for each plane.

We are interested in the stresses which deform the body, namely the stresses must not rotate the object. To prevent any rotation, stress σ_{ij} is equal to stress σ_{ji}, that is:

$$\sigma_{ij} = \sigma_{ji} \quad (2.3.1)$$

In order to prevent the rotation around the x_3 axis, the σ_{12} and σ_{21} must be balanced. Therefore, the stress tensor is symmetric as well as the strain tensor.

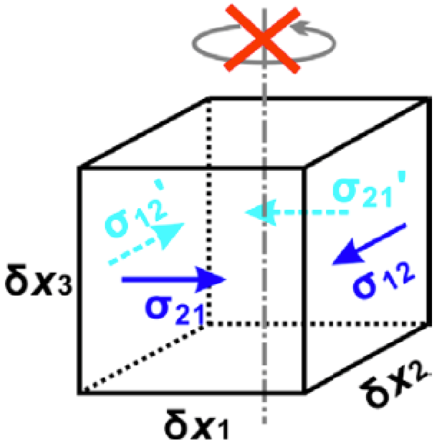

Fig. 2. The cubic prevents the rotation around x_3 axis. The blue stress tensor σ_{12} and σ_{21} must be balanced, therefore stress tensor σ_{12} become equal to σ_{21}.

The stress tensor can be expressed using the matrix expression as follows:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (2.3.2)$$

Since the stress tensor is a symmetric tensor (see eq. 2.3.1), it has six independent components. The components σ_{12} and σ_{21} are identical. Also, σ_{13} and σ_{31}, and σ_{23} and σ_{32} are identical, respectively. The signs of normal stresses will be positive in outwards and negative in inwards.

3.2 Principal stress

When only the normal stress acts on a certain surface, this normal stress refers to a principal stress. Like the strain tensor, the matrix of stress tensor is able to be diagonal. Because the stress matrix is symmetric the characteristic vectors are orthogonal to each other. Therefore, any combination of stresses can be decomposed with three normal stresses. The principal stress is expressed σ_{11}', σ_{22}', and σ_{33}', that is:

$$A^{-1}[\sigma_{ij}]A = \begin{bmatrix} \sigma_{11}' & 0 & 0 \\ 0 & \sigma_{22}' & 0 \\ 0 & 0 & \sigma_{33}' \end{bmatrix} \quad (2.3.3)$$

This shows the stress calculated in a coordinate system where the shear stress component become zero. For example, if we applied shear stress σ_{12} and σ_{21} to the body, like in the figure 3., these shear stresses can be regarded to the tensile stress in x_1' direction and the compressional stress in x_2' direction. These two directions are normal to each other. When we set the magnitude of the principal stress σ_{11}', σ_{22}', and σ_{33}' are $\sigma_{11}' > \sigma_{22}' > \sigma_{33}'$, they are called for a maximum principal stress, an intermediate principal stress, and a minimum principal stress, respectively. Also, the difference between the maximum principal stress and the minimum principal stress $(\sigma_{11}' - \sigma_{33}')$ is called differential stress.

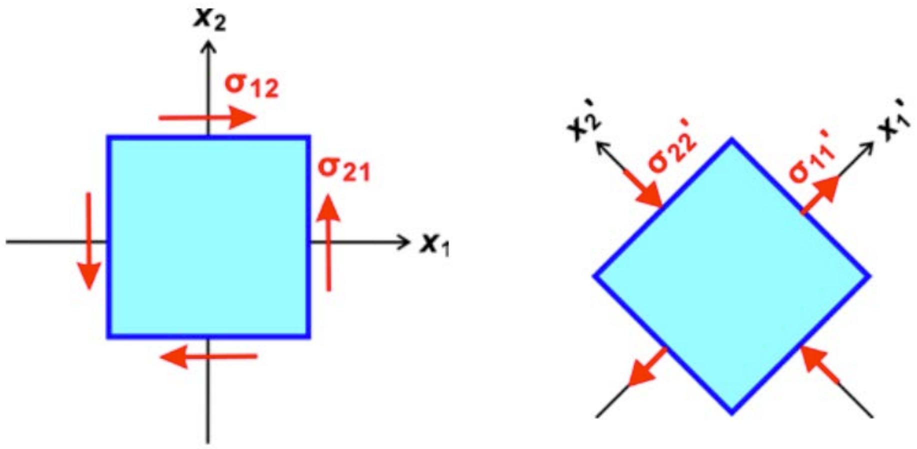

Fig. 3. Example of expressing shear stress as normal stress. This plane is normal to the x_3 axis. The shear stresses σ_{12} and σ_{21}, in the left figure, can be regarded to the tensile stress in $x_1{}'$ direction and the compressional stress in $x_2{}'$ direction, like in the right figure. Therefore, they can be considered as $\sigma_{11}{}'$ and $\sigma_{22}{}'$ respectively, and both tensors are orthogonal to each other.

A pressure (P) is the force that pushes normal to the surface of an object or any plane inside an object. The pressure is a minus average of the principal normal stresses $\sigma_{11}{}'$, $\sigma_{22}{}'$, and $\sigma_{33}{}'$, that is:

$$P = -(\sigma'_{11} + \sigma'_{22} + \sigma_{33}{}')/3 \qquad (2.3.4)$$

The positive pressure direction is inward because of the minus sign.

3.3 Simple shears and pure shears

We define two terms, a simple shear and a pure shear. Both terms are frequently used in the experimental works.

The simple shear is such that, parallel planes in a body remain parallel and maintain a constant distance but translates relative to each other. The simple shear deforms a circle into an ellipse, but in this case the main axis of strain rotates, unlike in pure shear. The stress tensor of simple shear is expressed as:

$$[\sigma_{ij}] = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (2.3.5)$$

There is only one non-zero component σ_{12}. This stress tensor is not symmetric because of the body rotates.

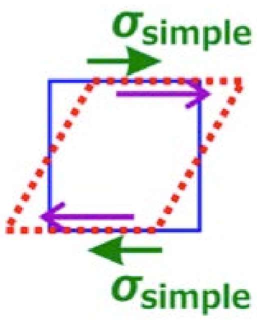

Fig. 4. Example of simple shear deformation. The blue square shows the original shape. The simple shear (σ_{simple}) deforms the shape into a parallelogram (red broken line).

The pure shear is the normal stress, which flattering the body without rotation. In the pure shear, the deformation occurs without changing the direction of the principal strain axis. The pure shear causes a circle to deform into an ellipse.

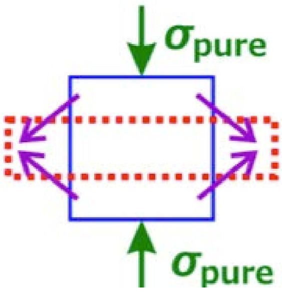

Fig. 5. Example of pure shear deformation. The blue square shows the original shape. The pure shear (σ_{pure}) deforms the shape into a rectangular (red broken line).