2. Elasticity

8. Elastic wave velocity of isotropic solids

8.1. 1D elastic wave velocity

Comprehending 1D elastic wave velocity of isotropic solids, which is the simplest model of elastic medium, is important for understanding elastic wave velocity of more complex model. Here we consider a one dimensional infinitesimal volume element between the point P and Q denoted by x and $x + \delta x$, respectively (Fig. 1). The mass of this volume element (m) is expressed as

$$m = \rho(x + \delta x - x) = \rho \delta x, \qquad (2.8.1)$$

where ρ is density of the volume. When this infinitesimal volume element is strained by a wave at time t, the point P and Q is displaced to the point P' denoted by $x + u(x,t)$ and Q' denoted by $x + \delta x + u(x + \delta x, t)$, respectively.

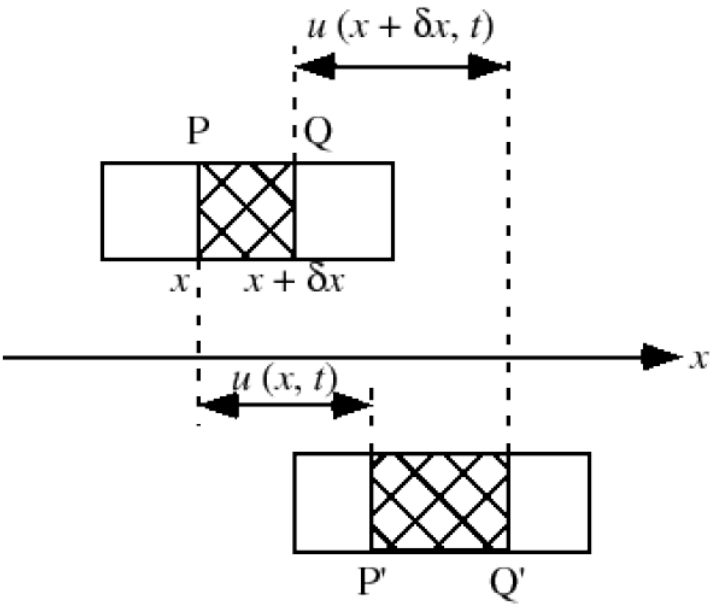

Fig. 1. The model of a one-dimensional infinitesimal volume element. The points P and Q are displaced to P' and Q' due to the elastic wave, respectively.

The force (F) acting to this strained volume element is obtained using Hooke's law by considering the stress. Since the strain of the minus side, the point of P', is $\frac{\partial u(x,t)}{\partial x}$, the stress ($\sigma$) on this side is

$$\sigma(x,t) = E\frac{\partial u(x,t)}{\partial x}, \qquad (2.8.2)$$

where E is the elastic constant. The stress on the plus side, the point of Q', is

$$\sigma(x + \delta x, t) = E\frac{\partial u(x + \delta x, t)}{\partial x}. \qquad (2.8.3)$$

Therefore, the net force to this infinitesimal element is expressed as

$$
\begin{aligned}
F &= \sigma(x+\delta x,t)-\sigma(x,t)\\
&= E\frac{\partial u(x+\delta x,t)}{\partial x}-E\frac{\partial u(x,t)}{\partial x}\\
&= E\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\delta x\\
&= E\frac{\partial^2 u}{\partial x^2}\delta x.
\end{aligned}
\tag{2.8.4}
$$

The elastic wave velocity is obtained by solving the equation of motion of the elastic element. The oscillation of an elastic body is explained based on the Newton's equation of motion:

$$F = m\frac{\partial^2 u}{\partial t^2}. \tag{2.8.5}$$

Substituting equation (2.8.1) and (2.8.4) to equation (2.8.5), the wave equation is obtained:

$$
\begin{aligned}
E\left(\frac{\partial^2 u}{\partial x^2}\right)\delta x &= \rho\delta x\frac{\partial^2 u}{\partial t^2},\\
\frac{\partial^2 u}{\partial t^2} &= \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}.
\end{aligned}
\tag{2.8.6}
$$

The solution of this wave function must have the following formula:

$$u = u_0\exp[ik(x-vt)], \tag{2.8.7}$$

where u_0, k, and v are amplitude, angular wave number, and velocity, respectively. Substituting this possible solution, the equation (2.8.7), into the wave equation (2.8.6), the left side is

$$\frac{\partial^2 u}{\partial t^2} = -k^2v^2u_0\exp[ik(x-vt)], \tag{2.8.8a}$$

and the right side is

$$\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = \frac{E}{\rho}\{-k^2u_0\exp[ik(x-vt)]\}. \tag{2.8.8b}$$

By equating these two formula, we obtain

$$k^2v^2u_0 = \frac{E}{\rho}(k^2u_0). \tag{2.8.9}$$

From this equation, we have the following formula:

$$v = \sqrt{\frac{E}{\rho}} \tag{2.8.10}$$

Thus, the velocity is proportional to the square root of the elastic constant and inversely proportional to the square root of the density.

8.2. 3D elastic wave velocities

In this section, we argue the wave velocities in a three-dimensional volume element (Fig. 2), which is essential for seismology and mineralogy. The edge lengths of this volume element are δx_1, δx_2, and δx_3. The point at the corner of this volume element in the direction to the original point is the point P denoted by (x_1, x_2, x_3). The point at the corner of this volume element in the opposite side is the point Q denoted by $(x_1+\delta x_1, x_2+\delta x_2, x_3+\delta x_3)$. We consider the elastic forces acting on this volume element in the direction of x_1. We first consider the forces on the front and rear faces where the areas are $\delta x_2\delta x_3$. The stresses on the front face and rear face in the direction of x_1 are $\sigma_{11}(x_1, x_2, x_3)$ and $\sigma_{11}(x_1+\delta x_1, x_2, x_3)$, respectively. Therefore, the net stress on the front and rear faces are

$\sigma_{11}(x_1+\delta x_1, x_2, x_3) - \sigma_{11}(x_1, x_2, x_3)$, which is equal to $\frac{\partial \sigma_{11}}{\partial x_1}\delta x_1$. Since the force is stress times area, the net force is $\left(\frac{\partial \sigma_{11}}{\partial x_1}\delta x_1\right)\delta x_2 \delta x_3$. In the same way, the net forces on the left and right faces and bottom and top faces are $\left(\frac{\partial \sigma_{12}}{\partial x_2}\delta x_2\right)\delta x_1 \delta x_3$ and $\left(\frac{\partial \sigma_{13}}{\partial x_3}\delta x_3\right)\delta x_1 \delta x_2$, respectively. Using these three formulas, therefore, the net force (F_{x_1}) in the x_1 direction is

$$\begin{aligned} F_{x_1} &= \left(\frac{\partial \sigma_{11}}{\partial x_1}\delta x_1\right)\delta x_2 \delta x_3 + \left(\frac{\partial \sigma_{12}}{\partial x_2}\delta x_2\right)\delta x_1 \delta x_3 + \left(\frac{\partial \sigma_{13}}{\partial x_3}\delta x_3\right)\delta x_1 \delta x_2 \\ &= \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}\right)\delta x_1 \delta x_2 \delta x_3. \end{aligned} \quad (2.8.11)$$

Since the mass of this element is $\rho \delta x_1 \delta x_2 \delta x_3$, where ρ is the density, the equation of motion in the x_1 direction is as follows:

$$\begin{aligned} \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}\right)\delta x_1 \delta x_2 \delta x_3 &= \rho \delta x_1 \delta x_2 \delta x_3 \frac{\partial^2 u_1}{\partial t^2} \\ \rho \frac{\partial^2 u_1}{\partial t^2} &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}. \end{aligned} \quad (2.8.12a)$$

Similarly, the equations of motion in the x_2 direction is

$$\rho \frac{\partial^2 u_2}{\partial t^2} = \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}, \quad (2.8.12b)$$

and in the x_3 direction is

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}. \quad (2.8.12c)$$

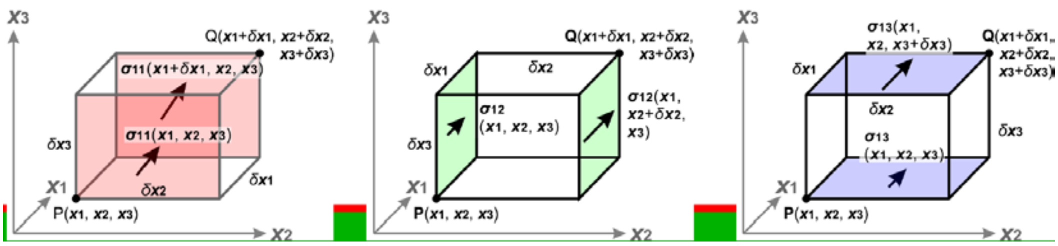

Fig. 2. The model of a three-dimensional infinitesimal volume element.

Next we introduce the three-dimensional equation of motion from the generalized Hooke's law of isotropic solids. The stresses acting this volume are σ_1 (the stress in the x_1 direction on the plane normal to the x_1 direction), σ_2 (the stress in the x_2 direction on the plane normal to the x_2 direction), σ_3 (the stress in the x_3 direction on the plane normal to the x_3 direction), σ_4 (the stress in the x_3 direction on the plane normal to the x_2 direction), σ_5 (the stress in the x_1 direction on the plane normal to the x_3 direction), and σ_6 (the stress in the x_2 direction on the plane normal to the x_1 direction). Based on Hooke's law ($\sigma_{ij} = \lambda \delta_{ij} \sum_k \varepsilon_{kk} + 2\mu\varepsilon_{ij}$, where λ and μ are Lame's constants), these stresses are expressed as follows:

$$\sigma_1 = \sigma_{11} = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2} + \lambda \frac{\partial u_3}{\partial x_3} \quad (2.8.13a)$$

$$\sigma_2 = \sigma_{22} = \lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3 = \lambda \frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu)\frac{\partial u_2}{\partial x_2} + \lambda \frac{\partial u_3}{\partial x_3} \quad (2.8.13b)$$

$$\sigma_3 = \sigma_{33} = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2\mu)\varepsilon_3 = \lambda\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3} \tag{2.8.13c}$$

$$\sigma_4 = \sigma_{23} = \sigma_{32} = 2\mu\varepsilon_4 = \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \tag{2.8.13d}$$

$$\sigma_5 = \sigma_{31} = \sigma_{13} = 2\mu\varepsilon_5 = \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \tag{2.8.13e}$$

$$\sigma_6 = \sigma_{12} = \sigma_{21} = 2\mu\varepsilon_6 = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{2.8.13f}$$

Substituting the equation (2.8.13a), (2.8.13f), and (2.8.13e) into the equation (2.8.12a), we obtain the equation of motion in the direction of x_1:

$$\begin{aligned}
\rho\frac{\partial^2 u_1}{\partial t^2} &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\
&= \frac{\partial}{\partial x_1}\left[(\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_3}{\partial x_3}\right] + \\
&\quad \frac{\partial}{\partial x_2}\left[\mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_3}\left[\mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\right] \\
&= (\lambda + \mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\lambda\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\lambda\frac{\partial u_3}{\partial x_3} + \\
&\quad \frac{\partial}{\partial x_2}\mu\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_2}\mu\frac{\partial u_2}{\partial x_1} + \frac{\partial}{\partial x_3}\mu\frac{\partial u_3}{\partial x_1} + \frac{\partial}{\partial x_3}\mu\frac{\partial u_1}{\partial x_3} \\
&= \mu\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + (\lambda + \mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \\
&\quad \mu\frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \mu\frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \mu\frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3} \\
&= \mu\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right) + (\lambda + \mu)\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right).
\end{aligned} \tag{2.8.14}$$

This equation can be simplified as:

$$\begin{aligned}
\rho\frac{\partial^2 u_1}{\partial t^2} &= \mu\left(\frac{\partial^2 u_1}{\partial {x_1}^2} + \frac{\partial^2 u_1}{\partial {x_2}^2} + \frac{\partial^2 u_1}{\partial {x_3}^2}\right) + (\lambda + \mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) \\
&= \mu\nabla^2 u_1 + (\lambda + \mu)\frac{\partial}{\partial x_1}\nabla\cdot\boldsymbol{u}
\end{aligned} \tag{2.8.15a}$$

Similarly, the equations of motion in the x_2 direction is

$$\rho\frac{\partial^2 u_2}{\partial t^2} = \mu\nabla^2 u_2 + (\lambda + \mu)\frac{\partial}{\partial x_2}\nabla\cdot\boldsymbol{u}, \tag{2.8.15b}$$

and in the x_3 direction is

$$\rho\frac{\partial^2 u_3}{\partial t^2} = \mu\nabla^2 u_3 + (\lambda + \mu)\frac{\partial}{\partial x_3}\nabla\cdot\boldsymbol{u}. \tag{2.8.15c}$$

Combining the three components (2.8.15a), (2.8.15b), and (2.8.15c), the equation of motion for three-dimensional elastic element is obtained:

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu\nabla^2\boldsymbol{u} + (\lambda + \mu)\nabla(\nabla\cdot\boldsymbol{u}) \tag{2.8.16}$$

The wave velocity for three-dimensional elastic volume is obtained from the equation of motion like one-dimensional model. The x_1 derivative of the equation (2.8.15a) is

$$\frac{\partial}{\partial x_1}\rho\frac{\partial^2}{\partial t^2}u_1 = \frac{\partial}{\partial x_1}\mu\nabla^2 u_1 + \frac{\partial}{\partial x_1}(\lambda+\mu)\frac{\partial}{\partial x_1}\nabla\cdot\boldsymbol{u}$$
$$\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_1}u_1 = \mu\nabla^2\frac{\partial}{\partial x_1}u_1 + (\lambda+\mu)\frac{\partial^2}{\partial {x_1}^2}\nabla\cdot\boldsymbol{u}. \tag{2.8.17a}$$

Similarly, the x_2 and x_3 derivatives of the equation (2.8.15b) and (2.8.15c), respectively, are

$$\rho\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_2}u_2 = \mu\nabla^2\frac{\partial}{\partial x_2}u_2 + (\lambda+\mu)\frac{\partial^2}{\partial {x_2}^2}\nabla\cdot\boldsymbol{u} \tag{2.8.17a}$$

$$\rho\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_3}u_3 = \mu\nabla^2\frac{\partial}{\partial x_3}u_3 + (\lambda+\mu)\frac{\partial^2}{\partial {x_3}^2}\nabla\cdot\boldsymbol{u}. \tag{2.8.17a}$$

The sum of the equation (2.8.17a), (2.8.17b), and (2.8.17c) is

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) = \mu\nabla^2\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) + (\lambda+\mu)\left(\frac{\partial^2}{\partial {x_1}^2}+\frac{\partial^2}{\partial {x_2}^2}+\frac{\partial^2}{\partial {x_3}^2}\right)\nabla\cdot\boldsymbol{u} \tag{2.8.18}$$

Considering $\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right) = \nabla\cdot\boldsymbol{u}$ and $\left(\frac{\partial^2}{\partial {x_1}^2}+\frac{\partial^2}{\partial {x_2}^2}+\frac{\partial^2}{\partial {x_3}^2}\right) = \nabla^2$, the equation (2.8.18) can be simplified as:

$$\rho\frac{\partial^2}{\partial t^2}(\nabla\cdot\boldsymbol{u}) = \mu\nabla^2(\nabla\cdot\boldsymbol{u}) + (\lambda+\mu)\nabla^2(\nabla\cdot\boldsymbol{u}) = (\lambda+2\mu)\nabla^2(\nabla\cdot\boldsymbol{u})$$
$$\frac{\partial^2}{\partial t^2}(\nabla\cdot\boldsymbol{u}) = \frac{\lambda+2\mu}{\rho}\nabla^2(\nabla\cdot\boldsymbol{u}) \tag{2.8.19}$$

which is the equation of wave and also can be introduced by calculating divergence of the equation (2.8.16). Since this wave is the propagation of $\nabla\cdot\boldsymbol{u} = \frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3} = \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$, the change of the volume, this equation explains the compressional wave or propagation of the volume change. This wave is often referred as "P-wave" in seismology, with the velocity of

$$v_p = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{K+\frac{4}{3}G}{\rho}} \tag{2.8.20}$$

where $K\ (=\lambda+\frac{2}{3}\mu)$ is the bulk modulus and $G\ (=\mu)$ is the rigidity or shear modulus. The x_3 derivative of (2.8.15b) is

$$\rho\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_3}u_2 = \mu\nabla^2\frac{\partial}{\partial x_3}u_2 + (\lambda+\mu)\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2}\nabla\cdot\boldsymbol{u}. \tag{2.8.21}$$

The x_2 derivative of (2.8.15c) is

$$\rho\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_2}u_3 = \mu\nabla^2\frac{\partial}{\partial x_2}u_3 + (\lambda+\mu)\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\nabla\cdot\boldsymbol{u}. \tag{2.8.22}$$

Subtracting (2.8.22) from (2.8.21), we have

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) = \mu\nabla^2\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right). \tag{2.8.23a}$$

Similar to the equation (2.8.23a), the following equations are obtained:

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\right) = \mu\nabla^2\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\right) \tag{2.8.23b}$$

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right)=\mu\nabla^2\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) \quad (2.8.23c)$$

These three equations can be combined and expressed using rotation as follows:

$$\frac{\partial^2}{\partial t^2}(\nabla\times\boldsymbol{u})=\frac{\mu}{\rho}\nabla^2(\nabla\times\boldsymbol{u}). \quad (2.8.24)$$

which is an equation of wave and also can be introduced by calculating rotation of the equation (2.8.16). Since this wave is the propagation of $\nabla\times\boldsymbol{u}$ (twist), this equation explains the shear wave, which is often referred as "S-wave" in seismology, with the velocity of

$$v_s=\sqrt{\frac{\mu}{\rho}}=\sqrt{\frac{G}{\rho}}. \quad (2.8.25)$$

It is clear from the equation (2.8.20) and (2.8.25) that P-wave propagates in the solid, liquid, and gas, but that S-wave can propagate only in the solid because liquid and gas have zero rigidity ($G=0$). Also we can see that the velocity of P-wave and S-wave are proportional to the square root of the elastic constant and inversely proportional to the square root of the density like the velocity of one-dimensional model (2.8.10).

8.3. Bulk sound velocity

Finally, we introduce the concept of the bulk sound velocity. Since liquid or gas have zero rigidity, no shear wave but only compressional wave propagate. The wave velocity of liquid or gas is

$$v_\Phi=\sqrt{\frac{K+\frac{4}{3}0}{\rho}}=\sqrt{\frac{K}{\rho}}. \quad (2.8.26)$$

which is bulk sound velocity. Let us apply this formula to solids, and this concept is referred as the bulk sound velocity. Note that there exists no "bulk sound" in the solids because rigidity is always greater than zero. This concept is introduced just for convenience because rigidity is more difficult to determine than bulk modulus. Bulk sound velocity can be calculated from seismic observations of P-wave velocity v_p and S-wave velocity v_s as

$$v_\Phi{}^2=v_p{}^2-\frac{4}{3}v_s{}^2. \quad (2.8.27)$$