2. Elasticity

9. Acoustic impedance

9.1 Wave reflection

Seismic wave is reflected by the earth interior when the structure is discontinuous. Typical examples of discontinuity are Mohorovičić discontinuity, 660-km discontinuity, and core-mantle boundary. If the acoustic impedance between two media is large, the wave is reflected. The acoustic impedance Z_S is defined as

$$Z_S = \rho v \tag{2.9.1}$$

where ρ is the density, v is the velocity. The definition of the acoustic impedance is explained in detail in section 9.4.

9.2 Energy of 1D wave

We derive the energy of 1D wave using the equation of wave in this section. The real part of the equation is

$$u = u_0 \cos(kx - \omega t) \tag{2.0.16'}$$

where u is the displacement, u_0 is the amplitude of displacement, k is the angular wave number, ω is the angular frequency. The partial derivative of displacement with respect to position x is

$$\frac{\partial u}{\partial x} = -k u_0 \sin(kx - \omega t) \tag{2.9.2}$$

The partial derivative of displacement with respect to time is

$$\frac{\partial u}{\partial t} = \omega u_0 \sin(kx - \omega t) \tag{2.9.3}$$

The potential (strain) energy, $U_P = \frac{1}{2} E \varepsilon^2$,

$$U_p = \frac{E}{2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{E k^2 u_0^2}{2} sin^2(kx - \omega t) = \frac{E k^2 u_0^2}{2} \frac{1 - cos2(kx - \omega t)}{2} \tag{2.9.4}$$

where E is the elastic constant, ε is the strain. The kinetic energy, $U_K = \frac{1}{2} \rho \dot{u}^2$

$$U_K = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 = \frac{\rho \omega^2 u_0^2}{2} sin^2(kx - \omega t) = \frac{\rho \omega^2 u_0^2}{2} \frac{1 - cos2(kx - \omega t)}{2} \tag{2.9.5}$$

The total energy is the sum of potential energy and kinetic energy, $U_T = U_P + U_P$

$$\begin{aligned} U_T &= \frac{E k^2 u_0^2}{2} \frac{1 - cos2(kx - \omega t)}{2} + \frac{\rho \omega^2 u_0^2}{2} \frac{1 - cos2(kx - \omega t)}{2} \\ &= \frac{1}{2}[E k^2 + \rho \omega^2] u_0^2 \frac{1 - cos2(kx - \omega t)}{2} \\ &= \frac{1}{2}\left[\frac{E}{\rho}\left(\frac{k}{\omega}\right)^2 + 1\right] \rho \omega^2 u_0^2 \frac{1 - cos2(kx - \omega t)}{2} \\ &= \frac{1}{2}\left[v^2 \left(\frac{1}{v}\right)^2 + 1\right] \rho \omega^2 u_0^2 \frac{1 - cos2(kx - \omega t)}{2} \\ &= \rho \omega^2 u_0^2 \frac{1 - cos2(kx - \omega t)}{2} \end{aligned} \tag{2.9.6}$$

where we use $v = f\lambda = \frac{\omega}{k}, v = \sqrt{\frac{E}{\rho}}$ to modify the equation. Figure 1 shows figure of seismic wave and energy density. At the node, the energy density is zero and at the anti-node, the energy density is highest.

The wavelength of energy density shows half of original wavelength. The average of energy is proportional to ρ, ω^2, u_0^2. The total energy is proportional to $\frac{1-cos2(kx-\omega t)}{2}$, which indicates that it is not kept constant within a given volume element. The energy density shows maxima at anti-node where $cos(kx-\omega t)=\pm 1$ or $cos(2kx-2\omega t)=-1$, and minima at node where $cos(kx-\omega t)=0$ or $cos(2kx-2\omega t)=+1$. Energy is transferred by the movement of nodes. It means a high energy density packet moves and transfers energy from one side to the other side. The transferred energy per unit time is proportional to the wave velocity, v

$$\rho\omega^2 u_0^2 \cdot \mathrm{v} \tag{2.9.7}$$

where energy in the one packet is proportional to $\rho\omega^2 u_0^2$.

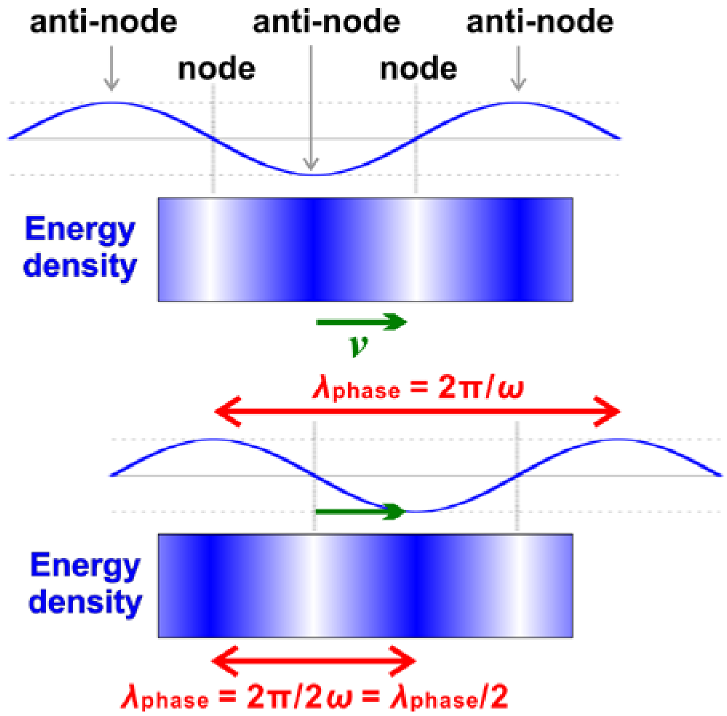

Fig. 1. Conceptional diagram of seismic wave and energy density. The blue curve shows the phase of the seismic wave. The green line shows wave velocity. The upper figure shows a certain situation, and the lower figure shows a subsequent situation.

9.3 Refection and transmission

We consider the elastic wave between two media. Figure 2 shows the setting of the elastic wave. There are two layers, the upper side is the Medium 1, and the lower side is the Medium 2. The incident wave comes from the upper side of the Medium 1 and reflects and transmits at the boundary between the Medium 1 and 2. The boundary is defined as $x=0$. X shows positive downwards. Then, we explain the notation of elastic wave. $u_i(x,t)$, $u_r(x,t)$, and $u_t(x,t)$ show the displacement by the incident, reflected, and transmitted wave at point x and time t, respectively. u_{i0}, u_{r0}, u_{t0} shows the amplitude of the incident, reflected, and transmitted wave, respectively. v_1, v_2 shows the velocity in the Medium 1 and 2, respectively.

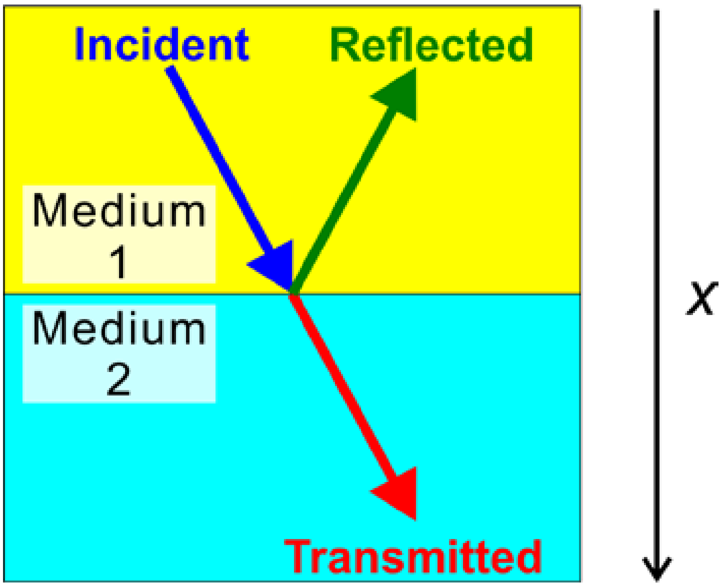

Fig. 2. Conceptional diagram of elastic wave between Medium 1 and 2. The bule line shows incident wave. The green line shows reflected wave. The red line shows transmitted wave. X axis shows positive downwards.

The displacements of both media at the boundary (x=0) must be equal:

$$u_i(0,\mathrm{t}) + u_r(0,\mathrm{t}) = u_t(0,\mathrm{t})$$
$$u_{i0} + u_{r0} = u_{t0} \tag{2.9.9}$$

Then, the conservation of energy is

$$\rho_1 \omega^2 u_{i0}^2 v_1 = \rho_1 \omega^2 u_{r0}^2 v_1 + \rho_2 \omega^2 u_{t0}^2 v_2 \tag{2.9.10}$$

because the frequency of the incident, reflected, and transmitted waves is common, and the energy of the incident wave delivers the energy of reflected and transmitted waves. By substituting equation (2.9.9) into equation (2.9.10), we have

$$\rho_1 u_{i0}^2 v_1 = \rho_1 u_{r0}^2 v_1 + \rho_2 (u_{i0} + u_{r0})^2 v_2 \tag{2.9.11}$$

By modifying equation (2.9.11), we have

$$\rho_1 v_1 u_{i0}^2 - \rho_1 v_1 u_{r0}^2 - \rho_2 v_2 u_{i0}^2 - 2\rho_2 v_2 u_{i0} u_{r0} - \rho_2 v_2 u_{r0}^2 = 0$$
$$\rho_1 v_1 u_{i0}^2 - \rho_2 v_2 u_{i0}^2 - u_{r0}\rho_1 v_1 u_{i0} u_{r0} - \rho_2 v_2 u_{i0} u_{r0} + u_{r0}\rho_1 v_1 u_{i0} u_{r0} - \rho_2 v_2 u_{i0} u_{r0}$$
$$- \rho_1 v_1 u_{r0}^2 - -\rho_2 v_2 u_{r0}^2 = 0$$
$$u_{i0}(\rho_1 v_1 u_{i0} - \rho_2 v_2 u_{i0} - \rho_1 v_1 u_{r0} - \rho_2 v_2 u_{r0})$$
$$+ u_{r0}(\rho_1 v_1 u_{i0} - \rho_2 v_2 u_{i0} - \rho_1 v_1 u_{r0} - \rho_2 v_2 u_{r0}) = 0$$
$$[u_{i0} + u_{r0}] \cdot [(\rho_1 v_1 - \rho_2 v_2) u_{i0} - (\rho_1 v_1 + \rho_2 v_2) u_{r0}] = 0 \tag{2.9.12}$$

The solution of the equation is $u_{i0} + u_{r0} = 0$ or $(\rho_1 v_1 - \rho_2 v_2) u_{i0} - (\rho_1 v_1 + \rho_2 v_2) u_{r0} = 0$. The first solution means complete reflection.

$$u_{i0} = -u_{r0} \tag{2.9.13}$$

The reflected wave has the same magnitudes of amplitude as but opposite phase to the incident wave. By substituting equation (2.9.13) into equation (2.9.9), we have

$$u_{t0} = u_{i0} + u_{r0} = 0 \tag{2.9.14}$$

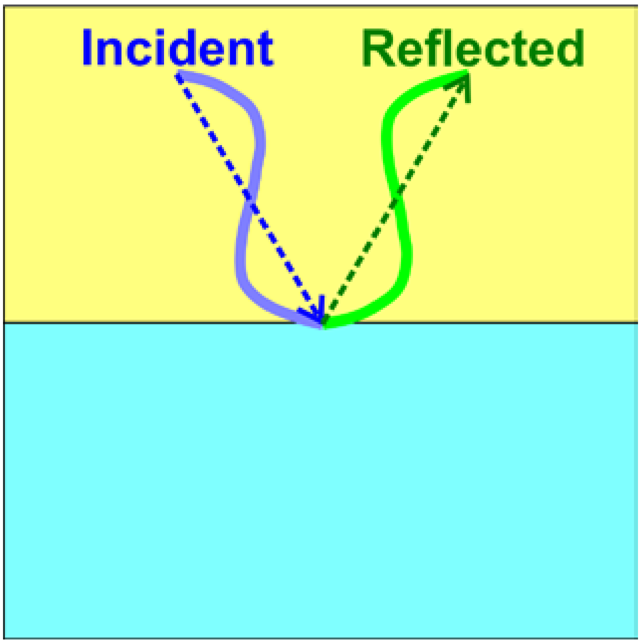

Fig. 3. Conceptional diagram of complete reflection of wave. The bule line shows incident wave. The green line shows reflected wave.

The equation means no transmitted wave and the reflected waves have the same magnitudes of amplitude as, but an opposite phase to the incident waves (Fig. 3). The second solution includes both reflection and transmission. It modifies

$$u_{r0} = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} u_{i0} \tag{2.9.15}$$

It means the amplitude of the reflected wave is proportional to the difference of ρv between two media, $\rho_1 v_1 - \rho_2 v_2$. If $\rho_1 v_1 < \rho_2 v_2$, it means anti-phase reflection, and medium 2 is stiffer and heavier as it is the "fixed end". The amplitude of the transmitted wave is,

$$u_{t0} = u_{i0} + \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} u_{i0} = \frac{2\rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2} u_{i0} \tag{2.9.16}$$

The incident and transmitted waves gave the same phases because $\rho_1, v_1, \rho_2, v_2 > 0$. If $\rho_1 v_1 = \rho_2 v_2$, $u_{t0} = u_{i0}$ from equation (2.9.16). It means 100% of transmissions.

9.4 Acoustic impedance

ρv is an essential parameter for reflectivity and transmissivity, and acoustic impedance is defined as $Z_S = \rho v$. We consider the reason why Z_S is called as "impedance". For example, impedance is the ratio of voltage to current in electricity. It means the ratio of the driving force to its result. Acoustic impedance is the ratio of the stress to the particle velocity. It means how fast the body part moves when we push the body. Particle velocity is

$$v_{particle} = \frac{\partial u}{\partial t} = \omega u_0 \sin(kx - \omega t) \tag{2.9.17}$$

when assuming the wave function: $u = u_0 \cos(kx - \omega t)$. The stress is

$$\sigma = E\varepsilon = E\frac{\partial u}{\partial x} = -Eku_0 \sin(kx - \omega t) \tag{2.9.18}$$

The acoustic impedance is

$$Z_S = \left|\frac{\sigma}{v_{particle}}\right| = \frac{Eku_0}{\omega u_0} = \frac{E}{\rho}\frac{k}{\omega}\rho = v_{phase}^2\frac{1}{v_{phase}}\rho = \rho v_{phase} \quad (2.9.19)$$

This is the reason why ρv is called as the acoustic impedance.

9.5 Reflectivity at the mantle discontinuities

We consider the reflectivity at the mantle discontinuities at 440-km and 660-km. First, we focus on the P-wave acoustic impedance of contrasts at 410-km discontinuity. The impedance at the upper part of 410-km,

$$Z_{S,P,410-} = 3.54\ \mathrm{g/cm^3} \times 8.91\ \mathrm{km/sec} = 3.15 \times 10^7\ \mathrm{kg/m^2sec}$$

The impedance at the lower part of 410-km,

$$Z_{S,P,410+} = 3.72\ \mathrm{g/cm^3} \times 9.13\ \mathrm{km/sec} = 3.40 \times 10^7\ \mathrm{kg/m^2sec}$$

The reflectivity at 410-km discontinuity,

$$R_{P,410} = \frac{3.15 \times 10^7 - 3.40 \times 10^7}{3.15 \times 10^7 + 3.40 \times 10^7} = -3.7\%$$

It means 3.7% reflection and overturn occurs.

Then, we focus on the P-wave acoustic impedance of contrasts at 660-km discontinuity. The impedance at the upper part of 660-km,

$$Z_{S,P,660-} = 3.99\ \mathrm{g/cm^3} \times 10.27\ \mathrm{km/sec} = 4.10 \times 10^7\ \mathrm{kg/m^2sec}$$

The impedance at the lower part of 660-km,

$$Z_{S,P,660+} = 4.38\ \mathrm{g/cm^3} \times 10.75\ \mathrm{km/sec} = 4.71 \times 10^7\ \mathrm{kg/m^2sec}$$

The reflectivity at 660-km discontinuity,

$$R_{P,410} = \frac{4.10 \times 10^7 - 4.71 \times 10^7}{4.10 \times 10^7 + 4.71 \times 10^7} = -6.9\%$$

It means 6.9% reflection and stronger discontinuity at 660-km than at 410-km.

9.6 Phase shift and overturn

We summarize the phase shift and overturn. No phase shift of reflection and transmission occurs, because the reflectivity and transmissivity are both real numbers. Overturn occurs when the incident from a low Z_S medium to high Z_S medium. In this case, there are negative reflectivity $\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} < 0$. From the shallower to the deeper part occur overturn because the shallower part has a lighter and lower velocity than the deeper part. Overturn does not occur when the incident goes from a high Z_S medium to a low Z_S medium. In this case, there are positive reflectivity $\frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} > 0$. It means upside reflection.

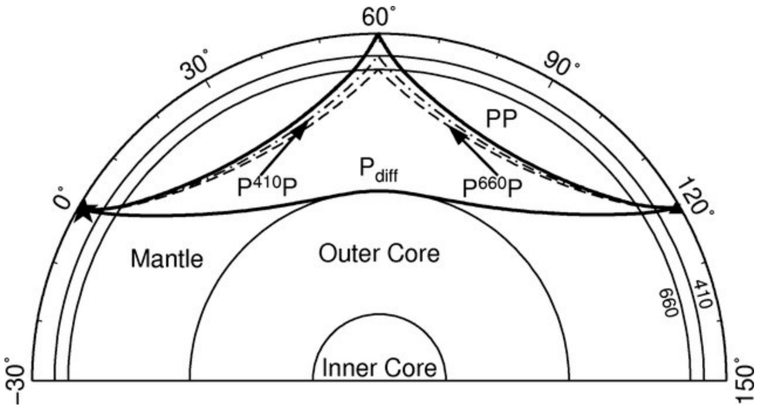

Fig. 4. Conceptional diagram of seismic wave in the earth interior includes mantle, outer core, and inner core. The black line shows seismic wave path.