2. Elasticity

2.8 Elastic wave velocity

2.8.1 1-D wave velocity

We first discuss elastic wave velocity from an example of a one-dimensional elastic element. 1D infinitesimal volume element is defined as $[x,\ x+\delta x]$ (Fig. 1). Points on the element are $P(x)$ and $Q(x+\delta x)$. Mass (m) between the $P(x)$ and $Q(x+\delta x)$ is:

$$m = \rho(x+\delta x - x) = \rho\delta x \tag{2.8.1}$$

ρ is one-dimensional density. Displacement of this element at a time (t) by a wave is expressed as $[u(x,t),\ u(x+\delta x,t)]$ when $P(x)$ goes to $P'(x+\delta x)$ and $Q(x)$ goes to $Q'(x+\delta x)$. We derive the forces acting to the infinitesimal volume element under strain $[x+u(x,t),\ x+\delta x+u(x+\delta x,t)]$ according to Hooke's law. $u(x,t)$ means the displacement at a point (x) and a time (t). On the minus side of the element, expressed by $P'(x+u(x,t))$, the stress $(\sigma(x,t))$ is given as follows:

$$\sigma(x,t) = E\frac{\partial u(x,t)}{\partial x} \tag{2.8.2}$$

E means elastic constant. On the plus side of the element, $Q'(x+\delta x+u(x+\delta x,t))$, the stress $\sigma(x+\delta x,t)$ at the point $(x+\delta x)$ is:

$$\sigma(x+\delta x,t) = E\frac{\partial u(x+\delta x,t)}{\partial x} \tag{2.8.3}$$

The net force (F) to the elastic element is calculated from the stresses acting on both sides of the elastic element:

$$\begin{aligned} F = \sigma(x+\delta x,t) - \sigma(x,t) &= E\frac{\partial u(x+\delta x,t)}{\partial x} - E\frac{\partial u(x,t)}{\partial x} = E\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\delta x \\ &= E\frac{\partial^2 u}{\partial x^2}\delta x \end{aligned} \tag{2.8.4}$$

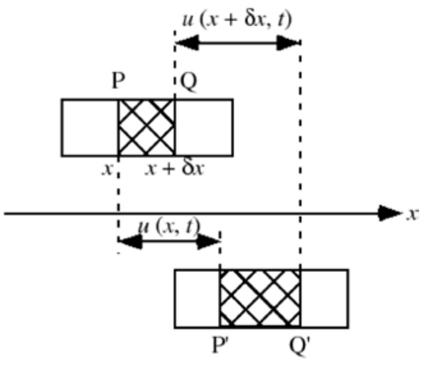

Fig. 1 1D elastic element

Using this formula (2.8.4), the equation of motion is:

$$E\frac{\partial^2 u}{\partial x^2}\delta x = F = m\frac{\partial^2 u}{\partial t^2} \tag{2.8.5}$$

The force and mass in (2.8.5) are substituted by (2.8.4) $F = E\frac{\partial^2 u}{\partial x^2}\delta x$ and (2.8.1) $m = \rho\delta x$, using

$E\frac{\partial^2 u}{\partial x^2}\delta x = \rho\delta x\frac{\partial^2 u}{\partial t^2}$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} \qquad (2.8.6)$$

This equation is the wave equation. The solution of the wave equation is given as follows:

$$u = u_0 \exp[ik(x - vt)] \qquad (2.8.7)$$

By substituting (2.8.7) into (2.8.6), left side of the equation $\frac{\partial^2 u}{\partial t^2} = -k^2 v^2 u_0 \exp[ik(x - vt)]$ and right side of the equation $\frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = -\frac{E}{\rho}k^2 u_0 \exp[ik(x - vt)]$ are obtained. By equating them, the wave velocity is obtained:

$$v = \sqrt{\frac{E}{\rho}} \qquad (2.8.8)$$

The wave velocity (v) is proportional to the square root of the elastic constant (E) and inversely to the density (ρ).

2.8.2 3-D wave velocities

We consider the example of a one-dimensional elastic element in the last section. Then, we discuss wave velocities of a three-dimensional elastic element. A 3D infinitesimal volume element is defined as $\delta x_1 \times \delta x_2 \times \delta x_3$. Using $F = E\frac{\partial^2 u}{\partial x^2}\delta x$, the elastic forces acting on the element in the x_1 direction:

$$\begin{aligned} F_{x_1} &= \left(\frac{\partial\sigma_{11}}{\partial x_1}\delta x_1\right)\delta x_2\delta x_3 + \left(\frac{\partial\sigma_{12}}{\partial x_2}\delta x_2\right)\delta x_1\delta x_3 + \left(\frac{\partial\sigma_{13}}{\partial x_3}\delta x_3\right)\delta x_1\delta x_2 \\ &= \left(\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3}\right)\delta x_1\delta x_2\delta x_3 \end{aligned} \qquad (2.8.9)$$

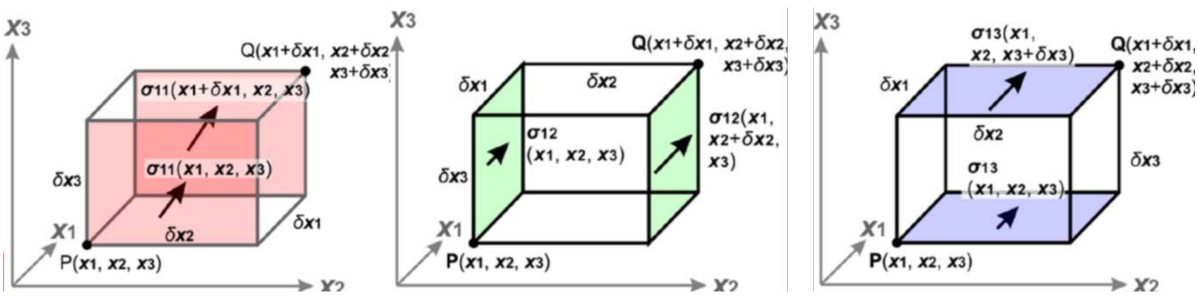

Fig. 2 3D elastic body

With combining (2.8.9), the equation of motion in the x_1 direction is expressed by using $F_{x_1} = m\frac{\partial^2 x_1}{\partial t^2}$ and $m = \rho\delta x_1\delta x_2\delta x_3$:

$$\begin{aligned} \left(\frac{\partial\sigma_{11}}{\partial x_1}\delta x_1 + \frac{\partial\sigma_{12}}{\partial x_1}\delta x_2 + \frac{\partial\sigma_{13}}{\partial x_1}\delta x_3\right)\delta x_1\delta x_2\delta x_3 &= \rho\delta x_1\delta x_2\delta x_3\frac{\partial^2 u_1}{\partial t^2} \\ \rho\frac{\partial^2 u_1}{\partial t^2} &= \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} \end{aligned} \qquad (2.8.10a)$$

Similarly, in the x_1 direction and x_1 direction,

$$\rho\frac{\partial^2 u_2}{\partial t^2}=\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3} \tag{2.8.10b}$$

$$\rho\frac{\partial^2 u_3}{\partial t^2}=\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3} \tag{2.8.10c}$$

The generalized Hooke's law of isotropic solids (2.5.31) using the Lame's constants λ and μ is defined as follows:

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix} \tag{2.5.31}$$

The stresses are defined as follows:

$$\sigma_1=\sigma_{11}=(\lambda+2\mu)\varepsilon_1+\lambda\varepsilon_2+\lambda\varepsilon_3=(\lambda+2\mu)\frac{\partial u_1}{\partial x_1}+\lambda\frac{\partial u_2}{\partial x_2}+\lambda\frac{\partial u_3}{\partial x_3} \tag{2.8.11a}$$

$$\sigma_2=\sigma_{22}=\lambda\varepsilon_1+(\lambda+2\mu)\varepsilon_2+\lambda\varepsilon_3=\lambda\frac{\partial u_1}{\partial x_1}+(\lambda+2\mu)\frac{\partial u_2}{\partial x_2}+\lambda\frac{\partial u_3}{\partial x_3} \tag{2.8.11b}$$

$$\sigma_3=\sigma_{33}=\lambda\varepsilon_1+\lambda\varepsilon_2+(\lambda+2\mu)\varepsilon_3=\lambda\frac{\partial u_1}{\partial x_1}+\lambda\frac{\partial u_2}{\partial x_2}+(\lambda+2\mu)\frac{\partial u_3}{\partial x_3} \tag{2.8.11c}$$

$$\sigma_3=\sigma_{33}=\lambda\varepsilon_1+\lambda\varepsilon_2+(\lambda+2\mu)\varepsilon_3=\lambda\frac{\partial u_1}{\partial x_1}+\lambda\frac{\partial u_2}{\partial x_2}+(\lambda+2\mu)\frac{\partial u_3}{\partial x_3} \tag{2.8.11d}$$

$$\sigma_4=\sigma_{23}=\sigma_{32}=2\mu\varepsilon_4=\mu(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}) \tag{2.8.11e}$$

$$\sigma_5=\sigma_{31}=\sigma_{13}=2\mu\varepsilon_5=\mu(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}) \tag{2.8.11f}$$

$$\sigma_6=\sigma_{12}=\sigma_{21}=2\mu\varepsilon_6=\mu(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}) \tag{2.8.11g}$$

Substituting (2.8.11a) $\sigma_{11}=(\lambda+2\mu)\varepsilon_1+\lambda\varepsilon_2+\lambda\varepsilon_3$, (2.5.11f) $\sigma_{12}=\mu(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1})$, (2.5.11e) $\sigma_{13}=\mu(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1})$, into (2.8.10a):

$$\rho\frac{\partial^2 u_1}{\partial t^2}=\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}$$

$$=\frac{\partial}{\partial x_1}\left[(\lambda+2\mu)\frac{\partial u_1}{\partial x_1}+\lambda\frac{\partial u_2}{\partial x_2}+\lambda\frac{\partial u_3}{\partial x_3}\right]+\frac{\partial}{\partial x_2}\left[\mu\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\right] \tag{2.8.12}$$

$$+\frac{\partial}{\partial x_3}\left[\mu\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\right]$$

$$= \mu \frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + (\lambda+\mu)\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \lambda \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3} + \mu \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \mu \frac{\partial}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \mu \frac{\partial}{\partial x_3}\frac{\partial u_3}{\partial x_1} + \mu \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}$$

$$= \mu \left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{\partial u_1}{\partial x_3}\right) + (\lambda+\mu)\left(\frac{\partial}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial}{\partial x_1}\frac{\partial u_3}{\partial x_3}\right)$$

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) = \mu \nabla^2 u_1 + (\lambda+\mu)\frac{\partial}{\partial x_1}\mathrm{div}(\boldsymbol{u}) \quad (2.8.13a)$$

Similarly,

$$\rho \frac{\partial^2 u_2}{\partial t^2} = \mu \nabla^2 u_2 + (\lambda+\mu)\frac{\partial}{\partial x_2}\mathrm{div}(\boldsymbol{u}) \quad (2.8.13b)$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \mu \nabla^2 u_3 + (\lambda+\mu)\frac{\partial}{\partial x_3}\mathrm{div}(\boldsymbol{u}) \quad (2.8.13c)$$

By combining (2.8.13a), (2.8.13b) and (2.8.13c),

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu \nabla^2 \boldsymbol{u} + (\lambda+\mu)\mathrm{grad}\big(\mathrm{div}(\boldsymbol{u})\big) = \mu \nabla^2 \boldsymbol{u} + (\lambda+\mu)\nabla^2 \boldsymbol{u} = (\lambda+2\mu)\nabla^2 \boldsymbol{u} \quad (2.8.14)$$

Compressional wave velocity is from (2.8.14):

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\nabla^2 \boldsymbol{u} \quad (2.8.14)$$

The components of the vector equation are as follows:

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) \quad (2.8.14a)$$

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right) \quad (2.8.14b)$$

$$\frac{\partial^2 u_3}{\partial t^2} = \frac{\lambda+2\mu}{\rho}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) \quad (2.8.14c)$$

This equation (2.8.14) is an equation of wave. The wave velocity is as follows using the relations of Lame’s constants, $K = \lambda + \frac{4}{3}\mu$ and $\mu = G$:

$$v_p = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{(K + \frac{4}{3}G)/\rho} \quad (2.8.15)$$

Here we derive the compressional wave velocity of the 3D elastic element. Then we derive the shear wave velocity from equations (2.8.14a), (2.8.14b) and (2.8.14c). First, we consider the shear component of wave in x_1 direction. The x_3 derivative of (2.8.13b)

$$\rho\frac{\partial^2 u_2}{\partial t^2} = \mu\nabla^2 u_2 + (\lambda+\mu)\frac{\partial}{\partial x_2}\mathrm{div}(\boldsymbol{u}) \quad \text{(the } x_2 \text{ component)}$$

$$\rho\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_3}u_2 = \mu\nabla^2\frac{\partial u_2}{\partial x_3} + (\lambda+\mu)\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\mathrm{div}(\boldsymbol{u}) \tag{2.8.16}$$

The x_2 derivative of (2.8.13c)

$$\rho\frac{\partial^2 u_3}{\partial t^2} = \mu\nabla^2 u_3 + (\lambda+\mu)\frac{\partial}{\partial x_3}\mathrm{div}(\boldsymbol{u}) \quad \text{(the } x_3 \text{ component)}$$

$$\rho\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x_2}u_3 = \mu\nabla^2\frac{\partial u_3}{\partial x_2} + (\lambda+\mu)\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\mathrm{div}(\boldsymbol{u}) \tag{2.8.17}$$

Subtracting (2.8.17) from (2.8.16), we have

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) = \mu\nabla^2\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) \tag{2.8.18a}$$

Similar to (2.8.18a),

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\right) = \mu\nabla^2\left(\frac{\partial u_3}{\partial x_1}-\frac{\partial u_1}{\partial x_3}\right) \tag{2.8.18b}$$

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) = \mu\nabla^2\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) \tag{2.8.18c}$$

From the definition of rotation $\mathrm{rot}(\boldsymbol{a}) \equiv \left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\quad \frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\quad \frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)$, we combine 3 components (2.8.18a), (2.8.18b) and (2.8.18c):

$$\frac{\partial^2}{\partial t^2}\mathrm{rot}(\boldsymbol{u}) = \frac{\mu}{\rho}\nabla^2\mathrm{rot}(\boldsymbol{u}) \tag{2.8.19}$$

An equation of wave $y = \mathrm{rot}(\boldsymbol{u})$ with the shear velocity:

$$v_s = \sqrt{\frac{\mu}{\rho}} = \sqrt{G/\rho} \tag{2.8.20}$$

2.8.3 Bulk sound velocity

Liquid or gas has zero rigidity. Therefore, no shear wave is observed in these materials, and only compressional wave exists in liquid or gas. The wave velocity in such material is:

$$v_\Phi = \sqrt{(K+\frac{4}{3}0)/\rho} = \sqrt{K/\rho} \tag{2.8.21}$$

Applying this formula to solids, bulk sound velocity (2.8.21) is obtained. Solid has non-zero rigidity. Therefore, bulk sound velocity is not observed in nature. Rigidity is more difficult to determine than bulk modulus because bulk modulus can be directly calculated from sound velocity observation. Bulk sound velocity v_Φ can be calculated from seismic observations of v_p and v_s.

$$v_\Phi = v_p^2 - \frac{4}{3}v_s^2 \tag{2.8.22}$$