2. Elasticity

5. Elastic Constants of symmetric solids

5.1 Symmetry of solids

Following Bravais classification, 32 symmetry groups of crystals subdivided into seven systems: the triclinic, the monoclinic, the orthorhombic, the tetragonal, the hexagonal, the trigonal and the cubic depending on unit-cell geometry.

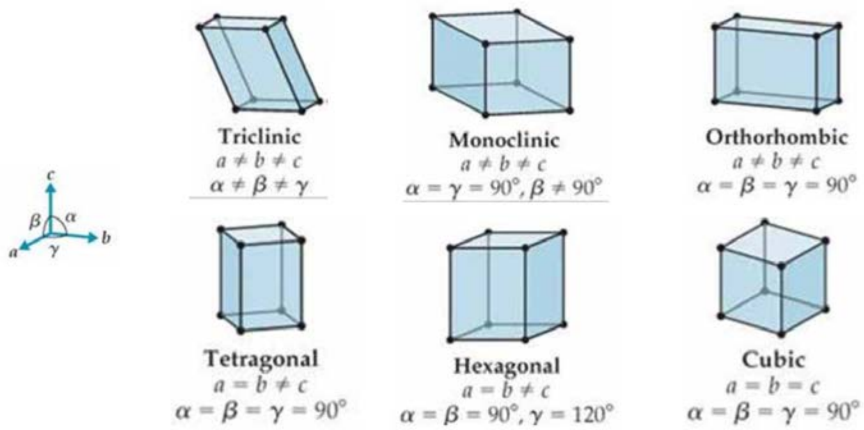

Fig. 1. Six symmetry groups by Bravais lattices: unit cells with cell parameters for each symmetry for primitive centering type.

Let us consider some patterns, which are characteristic of certain symmetry groups. Bravais unit cells of seven lattice systems are shown on figure 1. Note that the hexagonal system represents a six-fold symmetry, while the trigonal one represents three-fold symmetry relative to the c-axis. Concerning the cubic system, properties are not identical in the [100] (along a-axis) and [110] (equidistant from a&b axes) directions. If material has identical properties at any direction, it is called isotropic.

5.2 Deduction of elastic moduli

Each symmetry system is characterized by a certain elastic tensor C, which includes the different number of non-zero independent components. The number of such components depends on elastic behavior of the material. Given that in general case an elasticity tensor C consists of 21 independent components, this case is similar to the triclinic system. Considering the monoclinic system, the elasticity tensor C contains 13 independent constants; the orthorhombic, 9; the tetragonal; 6, the hexagonal and the trigonal, 5, the cubic, 3; and the isotropic, only 2 independent components.

Let us consider elasticity tensors for the all seven symmetry types.

5.3 Elastic constants in the triclinic system

In the triclinic symmetry system, all angles and edges are not equal to each other. Furthermore, no angles are equal to 90 degrees.

$$a \neq b \neq c, \alpha \neq 90°, \beta \neq 90°, \gamma \neq 90° \quad (2.5.1)$$

Therefore, in this case, an elasticity tensor has 21 non-zero independent components and in the Voight notation, it appears as following:

$$[C_{ij}]_{tric} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \quad (2.5.7)$$

To obtain elastic constants of all considered symmetry types, we should apply normal and shear strain to elementary cell of solids with a certain symmetry type, as is shown in the Figure 2.

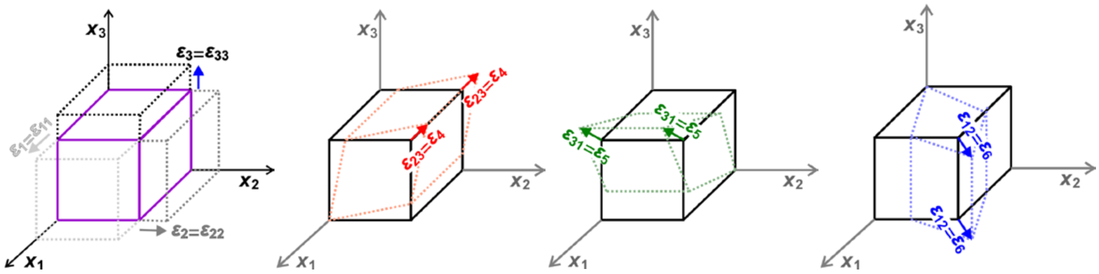

Fig. 2. Illustration of normal and shear strain applied to unit cell. Indexes are determined by fact that *i* denotes the direction along axis x_i, *j* denotes plane normal to axis x_j, single index is defined by Voight notation.

5.4 Elastic constants in the orthorhombic system

Now to understand elastic behavior and to obtain elastic constants (components of the elasticity tensor C) of material with the orthorhombic type of symmetry we need to apply normal and shear strain to an unit-cell volume of the solid.

The orthorhombic system is characterized by the following regularities (Fig. 3):

$$a \neq b \neq c, \alpha = \beta = \gamma = 90° \tag{2.5.2}$$

which means that the edges are not equal, while all angles are equal to 90 degrees. Figure 3 shows what happens when the unit-cell volume is exposed to strain.

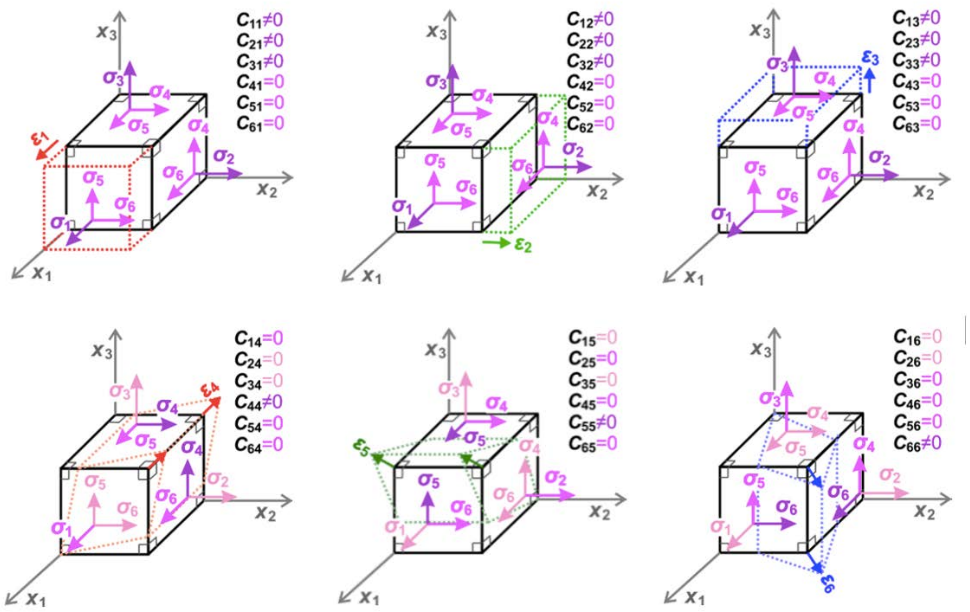

Fig. 3. Shear and normal strain applied to unit cell with the orthorhombic type of symmetry. *C* - elasticity tensor, ε - strain, σ - stress. Indexes of stress and strain defined by Voight notation.

We can see that there is no shear stress caused by normal strain, which leads to equality of the following components of the elasticity tensor to zero:

$$C_{41} = C_{51} = C_{61} = C_{42} = C_{52} = C_{62} = C_{43} = C_{53} = C_{63} = 0 \tag{2.5.14}$$

where Cij is the component of elasticity tensor, which connects σ_i and ε_j. In addition, the shear strain does not cause shear stresses normal to them, so that

$$C_{45} = C_{46} = C_{56} = 0 \tag{2.5.15}$$

In that way an elasticity tensor for the orthorhombic symmetry system contains nine non-zero independent components:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}_{tri} \Rightarrow \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}_{ortho} \tag{2.5.16}$$

5.5 Elastic constants in the monoclinic system

In the monoclinic system, only alpha and gamma angles are equal to 90 degrees, while beta angle is not equal to 90 degrees. Edges of the unit cell are not equal to each other.

$$\alpha = \gamma = 90°, \beta \neq 90°, usually\ a \neq b \neq c \tag{2.5.2}$$

Let us apply strain to the unit cell. Since beta angle is not equal to 90 degrees, the normal stress could change beta angle that will cause the shear stress σ_5. Thus, the following components of the elasticity tensor are non-zero:

$$C_{51} = C_{15} \neq 0,\ C_{52} = C_{25} \neq 0,\ C_{53} = C_{35} \neq 0 \tag{2.5.17}$$

In addition, due to the change in the beta angle by shear stress on the x_2-normal plane the following components will become non-zero:

$$C_{46} = C_{64} \neq 0 \tag{2.5.18}$$

Thus, the elasticity tensor of the monoclinic symmetry system contains 13 non-zero independent components:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}_{tri} \Rightarrow \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{22} & C_{23} & 0 & C_{25} & 0 \\ & & C_{33} & 0 & C_{35} & 0 \\ & & & C_{44} & 0 & C_{46} \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}_{mono} \tag{2.5.19}$$

5.6 Elastic constants in the tetragonal system

The tetragonal system is similar to the orthorhombic system except for $a = b$. Since crystal edge lengths with the tetragonal symmetry system are identical at two directions, the material has equal stress response at these directions. Thereby,

$$C_{11} = C_{22} \tag{2.5.8}$$

$$C_{44} = C_{55} \tag{2.5.9}$$

$$C_{23} = C_{13} \tag{2.5.10}$$

Note that the component C_{12} is not identical to C_{11} and C_{22}, while C_{12} is related to stress and strain at normal directions, and C_{11} and C_{22}, to stresses and strains at the same directions. Accordingly, an elasticity tensor of the tetragonal symmetry system contains 6 non-zero independent components:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}_{ortho} \Rightarrow \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix}_{tetra} \quad (2.5.11)$$

5.7 Elastic constants in the cubic system

The cubic system is similar to the orthorhombic system, but in this case all edges are identical (a=b=c). Therefore, the following expressions are valid:

$$C_{11} = C_{22} = C_{33} \quad (2.5.12)$$

$$C_{12} = C_{23} = C_{13} \quad (2.5.13)$$

$$C_{44} = C_{55} = C_{66} \quad (2.5.14)$$

Thus, the elasticity tensor of the cubic system contains three non-zero independent components:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}_{ortho} \Rightarrow \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}_{cubic} \quad (2.5.15)$$

5.8 Elastic constants of isotropic solids

The elastic properties for the isotropic material do not depend on direction, unlike the cubic system, in which elastic properties are not identical at [100] and [110] directions. To change from the cubic system to isotropic system we may consider two cases and try to find the conditions for identity of stresses in the following two cases (Fig. 4). The first case is the following: (1) apply ε_6 strain at [110] direction to the object with the cubic symmetry system; (2) rotate the object clockwise around the x_3 axis by 45 degrees. The second case is the following: (1) rotation of strain direction clockwise around the x_3 axis by 45 degrees; (2) combination of pure strain equivalent to shear stress ε_4.

Let us describe the first case in more detail.

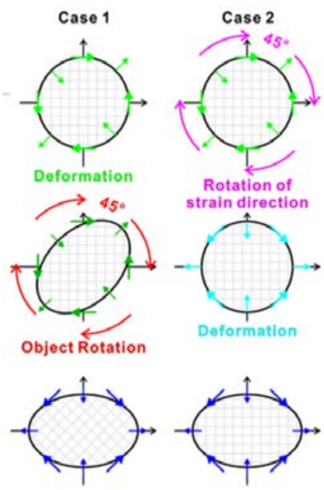

Fig. 4. Illustration of two cases, which show change from the cubic to isotropic system.

When we apply ε_6 strain at [110] direction to the object, the strain tensor is:

$$[\varepsilon] = \begin{bmatrix} 0 & \varepsilon_6 & 0 \\ \varepsilon_6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5.16}$$

The Hooke's law will be written as follows:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2\varepsilon_6 \end{bmatrix} \tag{2.5.17}$$

Accordingly, the stress tensor is equal to

$$[\sigma] = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix} = 2C_{44}\varepsilon_6 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5.20}$$

To rotate the object clockwise around x_3 axis by 45 degrees, the stress tensor must be multiplied by the rotation matrix [R]:

$$[R] = \begin{bmatrix} \cos(-\frac{\pi}{4}) & -sin(-\frac{\pi}{4}) & 0 \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.5.21}$$

Having done that, we get stress tensor [σ'] after rotation:

$$[\sigma'] = \begin{bmatrix} \sigma_1{}' & \sigma_6{}' & \sigma_5{}' \\ \sigma_6{}' & \sigma_2{}' & \sigma_4{}' \\ \sigma_5{}' & \sigma_4{}' & \sigma_3{}' \end{bmatrix} = [R][\sigma] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 2C_{44}\varepsilon_6 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \sqrt{2}C_{44}\varepsilon_6 \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5.22}$$

from which it follows that:

$$\sigma_1' = -\sigma_2' = \sigma_6' = \sqrt{2}C_{44}\varepsilon_6 \tag{2.5.23}$$

Now we will consider the second case in detail.

As in the previous case, to rotate the ε_6 strain clockwise around the x_3 axis by 45 degrees, the stress tensor must be multiplied by the rotation matrix [R]:

$$[\varepsilon'] = [R][\varepsilon] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \varepsilon_6 & 0 \\ \varepsilon_6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{\varepsilon_6}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5.24}$$

To calculate the stress tensor, we need to write the Hooke's law with rotated strain tensor [ε'] instead of [ε]:

$$[\sigma''] = [C]_{cubc}[\varepsilon'] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \frac{\varepsilon_6}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \tag{2.5.25}$$

In that way, the stress tensor is

$$[\sigma''] = [C][\varepsilon'] = \frac{\varepsilon_6}{\sqrt{2}}\begin{bmatrix} C_{11} - C_{12} \\ -C_{11} + C_{12} \\ 0 \\ 0 \\ 0 \\ 2C_{44} \end{bmatrix} \quad (2.5.25')$$

from which it follows:

$$\sigma_1'' = -\sigma_2'' = \frac{\varepsilon_6}{\sqrt{2}}(C_{11} - C_{12}) \quad (2.5.26)$$

$$\sigma_6'' = \sqrt{2}\varepsilon_6 C_{44} \quad (2.5.27)$$

Let us now compare the results of both cases.

For isotropic solids, σ_1' is equal to σ_1'', hence:

$$C_{11} - C_{22} = 2C_{44} \quad (2.5.28)$$

Following the expression above, C_{44} depends on C_{11} and C_{22}, which means that the elasticity tensor for isotropic system contains two non-zero independent components. Therefore, the elasticity tensor of the isotropic system is

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}_{iso} \quad (2.5.x)$$

where

$$C_{44} = \frac{1}{2}(C_{11} - C_{12}) \quad (2.5.x)$$

5.9 Lame's constants

For convenience, the elastic constants of the isotropic material are described by two values called the Lame parameters λ and μ. The Hooke's law, in terms of the Lame parameters, may be written as

$$\sigma_{ij} = \lambda\delta_{ij}\sum_k \varepsilon_{kk} + 2\mu\varepsilon_{ij} = \lambda\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{ij} \quad (2.5.29)$$

$$\delta_{ij}: \delta_{ij} = 1 \text{ if } i = j, \text{and}, \delta_{ij} = 0 \text{ if } i \neq j \text{ (Kronecker's delta)}$$

The stress tensor is

$$[\sigma_{ij}] = \begin{bmatrix} \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{11} & 2\mu\varepsilon_{12} & 2\mu\varepsilon_{13} \\ 2\mu\varepsilon_{21} & \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{22} & 2\mu\varepsilon_{23} \\ 2\mu\varepsilon_{31} & 2\mu\varepsilon_{32} & \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{33} \end{bmatrix} =$$

$$\begin{bmatrix} \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mu\varepsilon_1 & 2\mu\varepsilon_6 & 2\mu\varepsilon_5 \\ 2\mu\varepsilon_6 & \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mu\varepsilon_2 & 2\mu\varepsilon_4 \\ 2\mu\varepsilon_5 & 2\mu\varepsilon_4 & \lambda(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + 2\mu\varepsilon_3 \end{bmatrix} \quad 2.5.30$$

At the Voight notation, the matrix form is

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\2\varepsilon_4\\2\varepsilon_5\\2\varepsilon_6\end{bmatrix} \quad (2.5.31)$$

Thus, the relationships between elasticity tensor components and Lame parameters are

$$C_{11} = \lambda + 2\mu \quad (2.5.33)$$

$$C_{12} = \lambda \quad (2.5.34)$$

$$C_{44} = \mu \quad (2.5.35)$$

Note that the Lame parameter μ is a shear modulus or rigidity modulus, often denoted as G.

5.10 Consideration of the hexagonal, monoclinic and the trigonal symmetry systems

The hexagonal system represents the six-fold symmetry relative to x_3 (c) axis in comparison with the monoclinic system; in turn, the trigonal system represents the three-fold symmetry relative to the x_3 (c) axis, by contrast to the monoclinic system.

We will use the following algorithm to obtain an elastic tensor of the considered systems: (1) start from the monoclinic system; (2) replace the components with indices 2 and 5 by the components with indices 3 and 6, respectively. We need to do that, since in the hexagonal and the trigonal systems, the γ angle (between 1 (a) and 2 (b) directions) is not equal to 90 degrees, while for the monoclinic system, the β angle (between 1(a) and 3(c) directions) was not equal to 90 degrees.

$$\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0\\ & C_{22} & C_{23} & 0 & C_{25} & 0\\ & & C_{33} & 0 & C_{35} & 0\\ & & & C_{44} & 0 & C_{46}\\ & & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}_{\substack{mono\\ \beta\neq 90°}} \Rightarrow \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ & C_{22} & C_{23} & 0 & 0 & C_{26}\\ & & C_{33} & 0 & 0 & C_{36}\\ & & & C_{44} & C_{45} & 0\\ & & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}_{\substack{mono\\ \gamma\neq 90°}} \quad (2.5.36)$$

Since $a = b \neq c$, we can also replace indices 2 and 5 with 1 and 4, respectively; however we cannot replace indices of C_{12} and C_{45} components, because they characterized stress and strain at normal directions.

$$C_{22} = C_{11} \neq C_{12}, C_{23} = C_{13}, C_{23} = C_{16}, C_{55} = C_{44} \neq C_{45}$$

$$\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ & C_{22} & C_{23} & 0 & 0 & C_{26}\\ & & C_{33} & 0 & 0 & C_{36}\\ & & & C_{44} & C_{45} & 0\\ & & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}_{\substack{mono\\ \gamma\neq 90°\\ a\neq b\neq c}} \Rightarrow \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ & C_{11} & C_{13} & 0 & 0 & C_{16}\\ & & C_{33} & 0 & 0 & C_{36}\\ & & & C_{44} & C_{45} & 0\\ & & & & C_{44} & 0\\ & & & & & C_{66}\end{bmatrix}_{\substack{mono\\ \gamma\neq 90°\\ a=b\neq c}} \quad (2.5.37)$$

The next step in expression of an elasticity tensor of the hexagonal and the trigonal symmetry systems is an equalization of two cases of object rotation. As in the case of cubic – isotropic transition, we will consider the strain with subsequent rotation and rotation with further strain. Now we will rotate an object clockwise around x_3 axis by 60 degrees.

First, we apply strain to the object:

$$[\varepsilon] = \begin{bmatrix}\varepsilon_1 & \varepsilon_6 & 0\\ \varepsilon_6 & 0 & \varepsilon_4\\ 0 & \varepsilon_4 & \varepsilon_3\end{bmatrix} \quad (2.5.38)$$

Stress-strain relations in the symmetry

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ & C_{11} & C_{13} & 0 & 0 & C_{16}\\ & & C_{33} & 0 & 0 & C_{36}\\ & & & C_{44} & C_{45} & 0\\ & & & C_{45} & C_{44} & 0\\ & & & & & C_{66}\end{bmatrix}_{\substack{mono\\a=b\\\gamma\neq 90^\circ}}\begin{bmatrix}\varepsilon_1\\0\\\varepsilon_3\\2\varepsilon_4\\0\\2\varepsilon_6\end{bmatrix} \quad (2.5.39)$$

Therefore, the stress tensor is

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}C_{11}\varepsilon_1+C_{13}\varepsilon_3+2C_{16}\varepsilon_6\\C_{12}\varepsilon_1+C_{13}\varepsilon_3+2C_{16}\varepsilon_6\\C_{13}\varepsilon_1+C_{33}\varepsilon_3+2C_{36}\varepsilon_6\\2C_{44}\varepsilon_4\\2C_{45}\varepsilon_4\\C_{16}\varepsilon_1+C_{36}\varepsilon_3+2C_{66}\varepsilon_6\end{bmatrix} \quad (2.5.40)$$

$$\begin{bmatrix}\sigma_1 & \sigma_6 & \sigma_5\\\sigma_6 & \sigma_2 & \sigma_4\\\sigma_5 & \sigma_4 & \sigma_3\end{bmatrix} =\begin{bmatrix}C_{11}\varepsilon_1+C_{13}\varepsilon_3+2C_{16}\varepsilon_6 & C_{16}\varepsilon_1+C_{36}\varepsilon_3+2C_{66}\varepsilon_6 & 2C_{45}\varepsilon_4\\C_{16}\varepsilon_1+C_{36}\varepsilon_3+2C_{66}\varepsilon_6 & C_{12}\varepsilon_1+C_{13}\varepsilon_3+2C_{16}\varepsilon_6 & 2C_{44}\varepsilon_4\\2C_{45}\varepsilon_4 & 2C_{44}\varepsilon_4 & C_{13}\varepsilon_1+C_{33}\varepsilon_3+2C_{36}\varepsilon_6\end{bmatrix} \quad (2.5.41)$$

Then perform a rotation of the stress tensor clockwise around x_3 axis by 60 degrees. For that purpose, the stress tensor must be multiplied by the rotation matrix [R]:

$$[R]=\begin{bmatrix}\cos(-\frac{\pi}{3}) & -sin(-\frac{\pi}{3}) & 0\\\sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) & 0\\0 & 0 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & \sqrt{3} & 0\\-\sqrt{3} & 1 & 0\\0 & 0 & 2\end{bmatrix} \quad (2.5.42)$$

Rotation of the body with these stresses:

$$[\sigma']=[R][\sigma]=\frac{1}{2}\begin{bmatrix}1 & \sqrt{3} & 0\\-\sqrt{3} & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_1 & \sigma_6 & \sigma_5\\\sigma_6 & \sigma_2 & \sigma_4\\\sigma_5 & \sigma_4 & \sigma_3\end{bmatrix} =\frac{1}{2}\begin{bmatrix}\sigma_1+\sqrt{3}\sigma_6 & \sigma_6+\sqrt{3}\sigma_2 & \sigma_5+\sqrt{3}\sigma_4\\-\sqrt{3}\sigma_1+\sigma_6 & -\sqrt{3}\sigma_6+\sigma_2 & -\sqrt{3}\sigma_5+\sigma_4\\2\sigma_5 & 2\sigma_4 & 2\sigma_3\end{bmatrix} \quad (2.5.43)$$

By substituting (2.5.41) in (2.5.43), we get:

$$[\sigma']=\begin{bmatrix}\sigma'_1\\\sigma'_2\\\sigma'_3\\\sigma'_4\\\sigma'_5\\\sigma'_6\end{bmatrix}=\begin{bmatrix}\frac{(\sqrt{3}C_{66}+C_{16})\varepsilon_6+(\sqrt{3}C_{36}+C_{13})\varepsilon_3+(\sqrt{3}C_{16}+C_{11})\varepsilon_1}{2}\\\cdots\\\cdots\\\frac{\sigma'_{13}+\sigma'_{31}}{2}=(-\sqrt{3}C_{45}+3C_{44})\varepsilon_4\\\cdots\\\cdots\end{bmatrix} \quad (2.5.44)$$

In the second case, we first apply rotation:

$$[\varepsilon']=[R][\varepsilon]=\frac{1}{2}\begin{bmatrix}1 & \sqrt{3} & 0\\-\sqrt{3} & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\varepsilon_1 & \varepsilon_6 & 0\\\varepsilon_6 & 0 & \varepsilon_4\\0 & \varepsilon_4 & \varepsilon_3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\varepsilon_1+\sqrt{3}\varepsilon_6 & \varepsilon_6 & \sqrt{3}\varepsilon_4\\-\sqrt{3}\varepsilon_1+\varepsilon_6 & -\sqrt{3}\varepsilon_6 & \varepsilon_4\\0 & 2\varepsilon_4 & 2\varepsilon_3\end{bmatrix} \quad (2.5.45)$$

than let us write stresses caused by this strain:

$$[\sigma''] = [C][\varepsilon'] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{13} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{44} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 + \sqrt{3}\varepsilon_6 \\ -\sqrt{3}\varepsilon_6 \\ 2\varepsilon_3 \\ 2\varepsilon_4 \\ \sqrt{3}\varepsilon_4 \\ 2\varepsilon_6 - \sqrt{3}\varepsilon_1 \end{bmatrix} \quad (2.5.46)$$

This provides the following expression:

$$[\sigma''] = [C][\varepsilon'] = \begin{bmatrix} \frac{(\sqrt{3}C_{11} - \sqrt{3}C_{12} + 2C_{16})\varepsilon_6 + 2C_{13}\varepsilon_3 + (C_{11} - \sqrt{3}C_{16})\varepsilon_1}{2} \\ \cdots \\ \cdots \\ (\sqrt{3}C_{45} + 3C_{44})\varepsilon_4 \\ \cdots \\ \cdots \end{bmatrix} \quad (2.5.47)$$

The factors to each ε_i in σ'_j and σ''_j must be equal because $[\sigma'] = [\sigma'']$ should be held in any strain condition:

ε_1 in σ'_1 and σ''_1

$$\frac{1}{2}(\sqrt{3}C_{16} + C_{11}) = \frac{1}{2}(C_{11} - \sqrt{3}C_{16}) \Rightarrow C_{16} = 0 \quad (2.5.48)$$

ε_3 in σ'_1 and σ''_1

$$\frac{1}{2}(\sqrt{3}C_{36} + C_{13}) = \frac{1}{2}(2C_{13}) \Rightarrow C_{36} = 0 \quad (2.5.49)$$

ε_6 in σ'_1 and σ''_1

$$\frac{1}{2}(\sqrt{3}C_{66} + C_{16}) = \frac{1}{2}(\sqrt{3}C_{11} - \sqrt{3}C_{12} + 2C_{16}) \Rightarrow C_{66} = C_{11} - C_{12} \quad (2.5.50)$$

ε_4 in σ'_1 and σ''_1

$$-\sqrt{3}C_{45} + 3C_{44} = -(-\sqrt{3}C_{45} - 3C_{44}) \Rightarrow C_{45} = 0 \quad (2.5.51)$$

Thus, the tensor $[C_{ij}]$ of the hexagonal and the trigonal symmetry is

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix}_{\substack{Hexa \\ Trig}} \quad (2.5.37)$$

Let us compare this elasticity tensor with previous results. Note that the given systems have one independent component less than the tetragonal system ($C_{66} = C_{11} - C_{12}$). This means that the hexagonal and the trigonal systems are more “symmetric” than tetrahedral system. One more important thing is that C_{66} component has similar dependence as at transition from cubic to isotropic system ($2C_{44} = C_{11} - C_{12}$). This kind of relations seem to occur by equality of elasticity in non-axial directions.