2.4. Linear elasticity—Generalized Hooke's law

1. One-dimensional Hooke's law

Constitutive equations of solid materials within the Earth and the other planets are crucial to describe their elastic behavior, and subsequently discuss the propagation of seismic waves. If the deformation of a given solid material is uniaxial and its extent is small enough, its elasticity can be approximated by that of linear springs, *i.e.* Hooke's law. Considering a spring that is extended by x with the force F (Figure 1a), the F value should be proportional to the x value:

$$F = kx \tag{2.4.1}$$

where k is a constant that characterizes the stiffness of the spring. This relationship is called Hooke's law and can be applied to general elastic objects. For example, when a rod with length of L_0 and cross section of A_0 is extended by ΔL with the force F (Figure 1b), the strain ε and stress σ are respectively expressed as $\varepsilon = \Delta L/L_0$ and $\sigma = F/A_0$. In this case, Hooke's law can be expressed as below:

$$\sigma = C\varepsilon \tag{2.4.2}$$

where C is a constant that characterizes the stiffness of the rod.

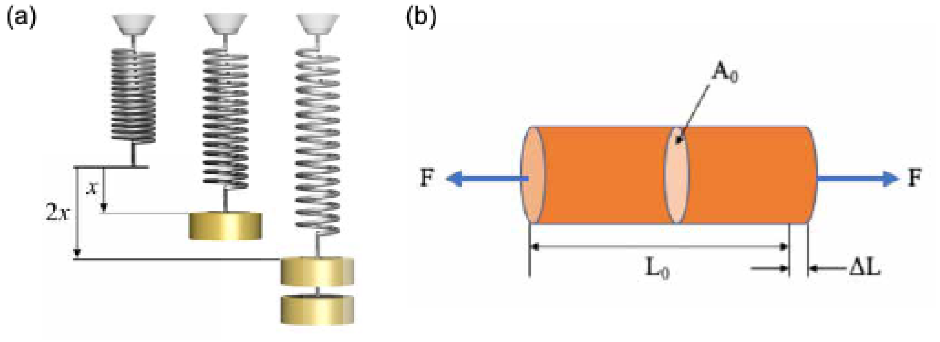

Fig. 1. Schematic diagrams of a linear spring and an extended rod. (a) spring that linearly extends to the weight mass; (b) elastic rod that extends in one direction.

2. Generalized Hooke's law

2.1 Fundamentals

To describe elasticity of solid materials that deform three-dimensionally, it is required to expand the one-dimensional Hooke's law to three-dimensional cases. Specifically, when a given elastic body is strained by ε_{ij}, not only σ_{ij} but also various stresses can occur. In other words, the stress σ_{ij} is generated not only by ε_{ij}, but other strains could contribute to it as well. Thus, by using a fourth-rank tensor, generalized Hooke's law is expressed as below:

$$\begin{aligned}\sigma_{ij} &= \Sigma_{kl} C_{ijkl}\varepsilon_{kl} \; (i,j,k,l = 1,2,3) \\ &= C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} \\ &\quad + C_{ij23}\varepsilon_{23} + C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}\end{aligned} \tag{2.4.3}$$

where C_{ijkl} is the stiffness tensor, or elasticity tensor. Combining the equations for all the stresses, one can obtain a matrix of the generalized Hooke's law:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_{21}\end{bmatrix}=\begin{bmatrix}C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321}\\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221}\\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{32}\\ \varepsilon_{31}\\ \varepsilon_{13}\\ \varepsilon_{12}\\ \varepsilon_{21}\end{bmatrix} \tag{2.4.4}$$

In addition, one can subsequently obtain the compliance as below:

$$\varepsilon_{ij} = \Sigma_{kl} S_{ijkl}\sigma_{kl}(i,j,k,l = 1,2,3) \tag{2.4.5}$$

where S_{ijkl} is the compliance. This is also a fourth-rank tensor and equivalent to the inverse of the stiffness tensor:

$$[S_{ijkl}] = [C_{ijkl}]^{-1} \tag{2.4.6}$$

Hence, as with the stiffness tensor, the compliance tensor can be expressed by using a matrix as below:

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{23}\\ \varepsilon_{32}\\ \varepsilon_{31}\\ \varepsilon_{13}\\ \varepsilon_{12}\\ \varepsilon_{21}\end{bmatrix}=\begin{bmatrix}S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1132} & S_{1131} & S_{1113} & S_{1112} & S_{1121}\\ S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2232} & S_{2231} & S_{2213} & S_{2212} & S_{2221}\\ S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3332} & S_{3331} & S_{3313} & S_{3312} & S_{3321}\\ S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2332} & S_{2331} & S_{2313} & S_{2312} & S_{2321}\\ S_{3211} & S_{3222} & S_{3233} & S_{3223} & S_{3232} & S_{3231} & S_{3213} & S_{3212} & S_{3221}\\ S_{3111} & S_{3122} & S_{3133} & S_{3123} & S_{3132} & S_{3131} & S_{3113} & S_{3112} & S_{3121}\\ S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1332} & S_{1331} & S_{1313} & S_{1312} & S_{1321}\\ S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1232} & S_{1231} & S_{1213} & S_{1212} & S_{1221}\\ S_{2111} & S_{2122} & S_{2133} & S_{2123} & S_{2132} & S_{2131} & S_{2113} & S_{2112} & S_{2121}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_{21}\end{bmatrix} \tag{2.4.7}$$

2.2 Symmetry of the stiffness tensor

Although the stiffness tensor has $3^4 = 81$ components, all of them are not necessarily independent with each other. Considering the condition where the object is never rotated, the following two relationships must be satisfied:

$$C_{ijkl} = C_{jikl} \quad \because \sigma_{ij} = \sigma_{ji} \tag{2.4.8}$$

$$C_{ijkl} = C_{ijlk} \quad \because \varepsilon_{kl} = \varepsilon_{lk} \tag{2.4.9}$$

Besides, symmetry of second derivatives of the strain energy leads to another constraint on the independency:

$$C_{ijkl} = C_{klij} \quad \sigma_{ij}/\varepsilon_{kl} = \sigma_{kl}/\varepsilon_{ij} \tag{2.4.10}$$

Summarizing, the components of the stiffness tensor must satisfy the following condition:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} = C_{klij} = C_{lkij} = C_{klji} = C_{lkji} \tag{2.4.11}$$

As a result, the number of the independent components of the stiffness tensor is calculated to be 21.

Here, by providing a proof of the relationship (2.4.10) and by demonstrating equivalent stiffness components, I try to explain the argument above in detail. First, because the work by the strains is stored within the elastic body as the strain energy, the following expression can be derived:

$$U=\int\sigma_{11}\,\mathrm{d}\varepsilon_{11}+\int\sigma_{12}\,\mathrm{d}\varepsilon_{12}+\int\sigma_{13}\,\mathrm{d}\varepsilon_{13}+\int\sigma_{21}\,\mathrm{d}\varepsilon_{21}+\int\sigma_{22}\,\mathrm{d}\varepsilon_{22}$$
$$+\int\sigma_{23}\,\mathrm{d}\varepsilon_{23}+\int\sigma_{31}\,\mathrm{d}\varepsilon_{31}+\int\sigma_{32}\,\mathrm{d}\varepsilon_{32}+\int\sigma_{33}\,\mathrm{d}\varepsilon_{33} \tag{2.4.12}$$

where U is the strain energy. Then, the total derivative of U is expressed as below:

$$\mathrm{d}U=\sigma_{11}\mathrm{d}\varepsilon_{11}+\sigma_{12}\mathrm{d}\varepsilon_{12}+\sigma_{13}\mathrm{d}\varepsilon_{13}+\sigma_{21}\mathrm{d}\varepsilon_{21}+\sigma_{22}\mathrm{d}\varepsilon_{22}$$
$$+\sigma_{23}\mathrm{d}\varepsilon_{23}+\sigma_{31}\mathrm{d}\varepsilon_{31}+\sigma_{32}\mathrm{d}\varepsilon_{32}+\sigma_{33}\mathrm{d}\varepsilon_{33} \tag{2.4.13}$$

Here it should be noted that this relationship is equivalent to the following:

$$\mathrm{d}U=\frac{\partial U}{\partial\varepsilon_{11}}\mathrm{d}\varepsilon_{11}+\frac{\partial U}{\partial\varepsilon_{12}}\mathrm{d}\varepsilon_{12}+\frac{\partial U}{\partial\varepsilon_{13}}\mathrm{d}\varepsilon_{13}+\frac{\partial U}{\partial\varepsilon_{21}}\mathrm{d}\varepsilon_{21}+\frac{\partial U}{\partial\varepsilon_{22}}\mathrm{d}\varepsilon_{22}$$
$$+\frac{\partial U}{\partial\varepsilon_{23}}\mathrm{d}\varepsilon_{23}+\frac{\partial U}{\partial\varepsilon_{31}}\mathrm{d}\varepsilon_{31}+\frac{\partial U}{\partial\varepsilon_{32}}\mathrm{d}\varepsilon_{32}+\frac{\partial U}{\partial\varepsilon_{33}}\mathrm{d}\varepsilon_{33} \tag{2.4.14}$$

Therefore, the relationship between the stress and the strain is expressed as below:

$$\sigma_{11}=\frac{\partial U}{\partial\varepsilon_{11}},\sigma_{12}=\frac{\partial U}{\partial\varepsilon_{12}},\sigma_{13}=\frac{\partial U}{\partial\varepsilon_{13}},\sigma_{21}=\frac{\partial U}{\partial\varepsilon_{21}},\sigma_{22}=\frac{\partial U}{\partial\varepsilon_{22}},$$
$$\sigma_{23}=\frac{\partial U}{\partial\varepsilon_{23}},\sigma_{31}=\frac{\partial U}{\partial\varepsilon_{31}},\sigma_{32}=\frac{\partial U}{\partial\varepsilon_{32}},\sigma_{33}=\frac{\partial U}{\partial\varepsilon_{33}} \tag{2.4.15}$$

Applying the generalized Hooke's law (2.4.3), the following equations are obtained:

$$\frac{\partial U}{\partial\varepsilon_{ij}}=\sigma_{ij}=C_{ij11}\varepsilon_{11}+C_{ij12}\varepsilon_{12}+C_{ij13}\varepsilon_{13}+C_{ij21}\varepsilon_{21}+C_{ij22}\varepsilon_{22}$$
$$+C_{ij23}\varepsilon_{23}+C_{ij31}\varepsilon_{31}+C_{ij32}\varepsilon_{32}+C_{ij33}\varepsilon_{33} \tag{2.4.16}$$

$$\frac{\partial^2 U}{\partial\varepsilon_{kl}\,\partial\varepsilon_{ij}}=C_{ijkl}\frac{\mathrm{d}\varepsilon_{kl}}{\mathrm{d}\varepsilon_{kl}}=C_{ijkl} \tag{2.4.17}$$

Note that the equation (2.4.17) is valid because all the strains are independent with each other, and because strains other than ε_{kl} disappear by the differentiation with respect to ε_{kl}. This clarifies that the elastic constant is equivalent to the second derivative of the strain energy with respect to the strains. Since the order of taking partial derivatives is exchangeable, the relationship (2.4.10) is obtained.

Finally, considering the equation (2.4.11), I demonstrate that the number of the independent components of the stiffness tensor is 21. Based on the non-rotation condition, the 36 individual non-equivalent components in the stiffness matrix are expressed with different colors (and styles) as below:

C_{1111} C_{1122} C_{1133}
C_{2211} C_{2222} C_{2233}
C_{3311} C_{3322} C_{3333}

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{32}\\\sigma_{31}\\\sigma_{13}\\\sigma_{12}\\\sigma_{21}\end{bmatrix}=\begin{bmatrix} & & & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121}\\ & & & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221}\\ & & & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321}\\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221}\\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{32}\\\varepsilon_{31}\\\varepsilon_{13}\\\varepsilon_{12}\\\varepsilon_{21}\end{bmatrix} \tag{2.4.4}$$

The individual components are: (1) C_{1111} (black with yellow outlines); (2) C_{1122} (black with green outlines); (3) C_{1133} (black with cyan outlines); (4) C_{1123} and C_{1132} (gray with yellow shading); (5) C_{1131} and C_{1113} (gray with magenta shading); (6) C_{1112} and C_{1121} (gray with violet shading); (7) C_{2211} (black with purple outline); (8) C_{2222} (black with navy outline); (9) C_{2233} (black with red outline); (10) C_{2223} and C_{2232} (gray with lime shading); (11) C_{2231} and C_{2213} (gray with blue shading); (12) C_{2212} and C_{2221} (gray with turquoise shading); (13) C_{3311} (black with burgundy outline); (14) C_{3322} (black with lime outline); (15) C_{3333} (black with orange outline); (16) C_{3323} and C_{3332} (gray with cyan shading); (17) C_{3331} and C_{3313} (gray with red shading); (18) C_{3312} and C_{3321} (gray with dark-green shading); (19) C_{2311} and C_{3211} (black with yellow shading); (20) C_{2322} and C_{3222} (black with lime shading); (21) C_{2333} and C_{3233} (black with cyan shading); (22) C_{2323}, C_{2332}, C_{3223} and C_{3232} (yellow); (23) C_{2331}, C_{2313}, C_{3231} and C_{3213} (green); (24) C_{2312}, C_{2321}, C_{3212} and C_{3221} (cyan); (25) C_{3111} and C_{1311} (black with magenta shading); (26) C_{3122} and C_{1322} (black with blue shading); (27) C_{3133} and C_{1333} (black with red shading); (28) C_{3123}, C_{3132}, C_{1323} and C_{1332} (violet); (29) C_{3131}, C_{3113}, C_{1331} and C_{1313} (navy); (30) C_{3112}, C_{3121}, C_{1312} and C_{1321} (red); (31) C_{1211} and C_{2111} (black with violet shading); (32) C_{1222} and C_{2122} (black with turquoise shading); (33) C_{1233} and C_{2133} (black with dark-green shading); (34) C_{1223}, C_{1232}, C_{2123} and C_{2132} (burgundy); (35) C_{1231}, C_{1213}, C_{2131} and C_{2113} (lime); (36) C_{1212}, C_{1221}, C_{2112} and C_{2121} (orange). Note that this symmetry is derived from the following two relationships. Focusing on the equation (2.4.8), which means the symmetry of shear stresses, the stiffness matrix is characterized by the symmetry depicted in the matrix below:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix} \quad (2.4.4)$$

The stiffness components aligned vertically within the same color group are equivalent to each other. In addition, focusing on the equation (2.4.9), which means the symmetry of shear strains, the stiffness matrix is characterized by the symmetry depicted in the matrix below:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix} \quad (2.4.4)$$

In this case, the stiffness components aligned horizontally within the same color group are equivalent to each other. Then, additionally considering the condition of (2.4.10), one can obtain 21 independent components:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{21} \end{bmatrix} = \begin{bmatrix} & & & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121} \\ & & & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221} \\ & & & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix} \quad (2.4.4)$$

The individual non-equivalent components are: (1') C_{1111} (black with yellow outlines); (2') C_{1122} and C_{2211} (black with purple outlines); (3') C_{1133} and C_{3311} (black with cyan outlines); (4') C_{1123},

C_{1132}, C_{2311} and C_{3211} (black with yellow shading); (5') C_{1131}, C_{1113}, C_{3111} and C_{1311} (black with magenta shading); (6') C_{1112}, C_{1121}, C_{1211} and C_{2111} (black with violet shading); (7') C_{2222} (black with navy outlines); (8') C_{2233} and C_{3322} (black with red outlines); (9') C_{2223}, C_{2232}, C_{2322} and C_{3222} (black with lime shading); (10') C_{2231}, C_{2213}, C_{3122} and C_{1322} (black with blue shading); (11') C_{2212}, C_{2221}, C_{1222} and C_{2122} (black with turquoise shading); (12') C_{3333} (black with orange outlines); (13') C_{3323}, C_{3332}, C_{2333} and C_{3233} (black with cyan shading); (14') C_{3331}, C_{3313}, C_{3133} and C_{1333} (black with red shading); (15') C_{3312}, C_{3321}, C_{1233} and C_{2133} (black with dark-green shading); (16') C_{2323}, C_{2332}, C_{3223} and C_{3232} (yellow); (17') C_{2331}, C_{2313}, C_{3231}, C_{3213}, C_{3123}, C_{3132}, C_{1323} and C_{1332} (purple); (18') C_{2312}, C_{2321}, C_{3212}, C_{3221}, C_{1223}, C_{1232}, C_{2123} and C_{2132} (cyan); (19') C_{3131}, C_{3113}, C_{1331} and C_{1313} (navy); (20') C_{3112}, C_{3121}, C_{1312}, C_{1321}, C_{1231}, C_{1213}, C_{2131} and C_{2113} (red); and (21') C_{1212}, C_{1221}, C_{2112} and C_{2121} (orange). This is consistent with the number of the newly added symmetric conditions: 36 – 21 = 15 [(2) = (7), (3) = (13), (4) = (19), (5) = (25), (6) = (31), (9) = (14), (10) = (20), (11) = (26), (12) = (32), (16) = (21), (17) = (27), (18) = (33), (23) = (28), (24) = (34), and (30) = (35)].

2.3 Stiffness tensor with Voigt notation

As mentioned above, since the strain and stress are second-rank tensors, the stiffness and compliance matrices are subsequently expressed by much complicated, fourth-rank tensors. However, focusing on the symmetry of strains and stresses, these notations can be converted into simpler ones. Specifically, Voigt notation is useful to describe such complicated components. In the Voigt notation, subscripts of the individual components are converted in the following manner: 11 → 1; 22 → 2; 33 → 3; 23, 32 → 4; 13, 31 → 5; and 12, 21 → 6. Therefore, stresses can be expressed as below (Figure 2a):

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix} \tag{2.4.18}$$

Similarly, strains are expressed as below (Figure 2b):

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_6 & \varepsilon_5 \\ \varepsilon_6 & \varepsilon_2 & \varepsilon_4 \\ \varepsilon_5 & \varepsilon_4 & \varepsilon_3 \end{bmatrix} \tag{2.4.19}$$

Finally, the generalized Hooke's law is expressed as below:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix} \tag{2.4.20}$$

Notably, in this notation, the factors of '2' appear in the strain tensor. This is because the subscripts of 4 (or 5 and 6) include two complementary directions of shear strains or stresses. For example, considering only one shear strain, the original notation of the generalized Hooke's law is expressed as below:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{13}\\ \sigma_{12}\\ \sigma_{21}\end{bmatrix}=\begin{bmatrix} & & & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121}\\ & & & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221}\\ & & & C_{3323} & C_{3332} & C_{3331} & C_{3313} & C_{3312} & C_{3321}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321}\\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3232} & C_{3231} & C_{3213} & C_{3212} & C_{3221}\\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3132} & C_{3131} & C_{3113} & C_{3112} & C_{3121}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1332} & C_{1331} & C_{1313} & C_{1312} & C_{1321}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1232} & C_{1231} & C_{1213} & C_{1212} & C_{1221}\\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121}\end{bmatrix}\begin{bmatrix}0\\0\\0\\ \varepsilon_{23}\\ \varepsilon_{32}\\0\\0\\0\\0\end{bmatrix} \quad (2.4.21)$$

Rearranging,

$$\sigma_{ij} = C_{ij23}\varepsilon_{23} + C_{ij32}\varepsilon_{32} = C_{k4}\varepsilon_4 + C_{k4}\varepsilon_4 = 2C_{k4}\varepsilon_4 \quad (2.4.22)$$

where the subscript *k* is a dummy variable that indicates the subscript of the stress in the Voigt notation equivalent to *ij* in the original notation. On the other hand, the generalized Hooke's law with the Voigt notation is expressed as below:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix}=\begin{bmatrix} & & & C_{14} & C_{15} & C_{16}\\ & & & C_{24} & C_{25} & C_{26}\\ & & & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}0\\0\\0\\ a\varepsilon_4\\0\\0\end{bmatrix} \quad (2.4.23)$$

Rearranging,

$$\sigma_i = aC_{i4}\varepsilon_4 \quad (2.4.24)$$

Thus, from (2.4.22) and (2.4.24), *a* must equals 2 for the consistency.

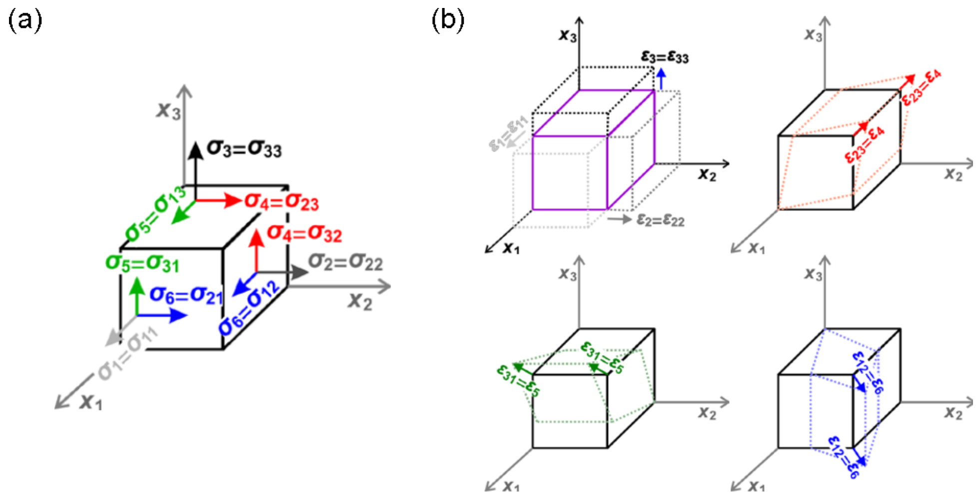

Fig. 2. Schematic diagrams of Voigt notation. (a) Voigt notation of stresses; (b) Voigt notation of strains.