2. Elasticity

3. Stress

3.1 Stress tensor

Stress is the force per unit area, applied across a small boundary. The boundary can be defined on the surface and anywhere within the body. The stress is the measure of a function to deform the body.

For describing stress state, each component should be related to two directions. One is related with the direction of the vector itself and the other is related to the boundary where it is applied. Figure 1 shown the 9 components of stress tensor in a tridimensional body. First subindex indicates the direction of the stress and second is related to the face where it is applied. Stress component in the x_i direction across the boundary normal to the x_j direction can be represented by σ_{ij}.

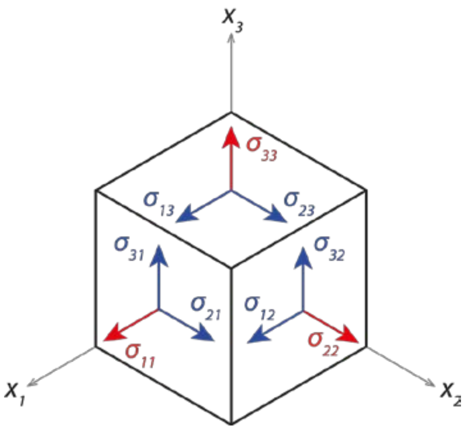

Fig. 1. Tridimensional body with a stress tensor in their positive faces with respect to three arbitrary axes. Arrows indicate the direction of the stress and where it is applied. Subindexes describe the direction and position where stress is applied.

The stress is a tensor of second rank because each magnitude of its components requires two directions to be completely specified. The way components show direction is using indexes, as was previously described. From this definition, there are two kind of stresses that can be applied on a surface: normal stress, which indexes are $i = j$, acting normal to the boundary; and shear stress, where $i \neq j$, and are acting parallel to the boundary.

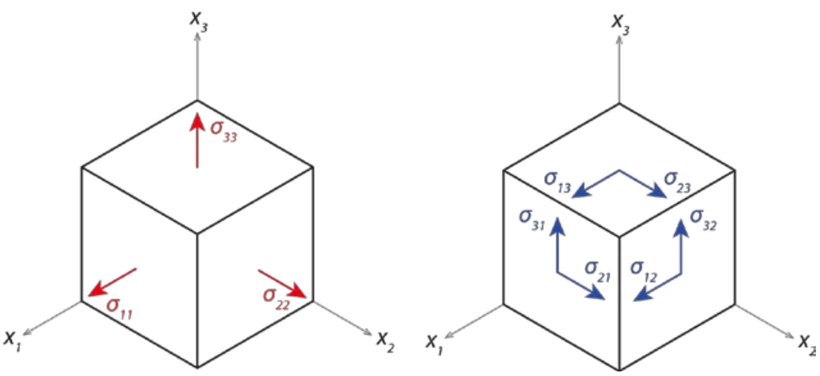

Fig. 2. Stresses nomenclature for each face. In red (left) are the normal stresses, indicated by same indexes (11, 22 and 33) and in blue (right) the shear stresses.

The cube used for stress concept, by definition, is in equilibrium. It is that is not translating nor rotating. In this sense, some of the shear components, are operating in opposite directions. If one of the components is larger than the other, the cube will rotate (left side of Figure 3). For that reason, these shear components require to be equal,

$$\sigma_{ij} = \sigma_{ji} \tag{2.3.1}$$

Considering same thinking for the rest of the components, the independent components of the stress tensor are reduced from nine to six.

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.3.2}$$

where all σ_{ij} represent the components of the stress tensor.

From the symmetry,

$$\sigma_{21} = \sigma_{12}$$
$$\sigma_{31} = \sigma_{13}$$
$$\sigma_{23} = \sigma_{32}$$

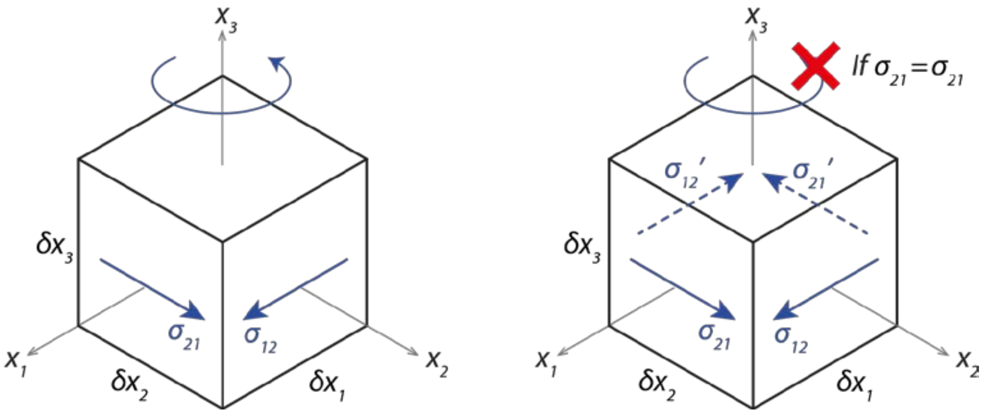

Fig. 3. Shear stresses related to rotation tendency with respect of axis x_3 in figure left. At right side, equilibrium conditions require that both shear stresses must have same magnitude. Furthermore, in the opposite sides must be the same condition to be sure about the equilibrium condition.

The signs of normal stress are positive when it is outwards and negative when it is inwards (red arrows in Figure 1).

3.2 Principal stresses

The stress tensor can be orthogonalized, and the characteristic vectors are orthogonal due to the symmetricity. This can be written as shown in the equation 2.3.3.

$$A^{-1}[\sigma_{ij}]A = \begin{bmatrix} \sigma'_{11} & 0 & 0 \\ 0 & \sigma'_{22} & 0 \\ 0 & 0 & \sigma'_{33} \end{bmatrix} \tag{2.3.3}$$

where σ'_{11}, σ'_{22} and σ'_{33} represent the principal stresses of a given stress tensor.

Geometrically, the principal stresses can be calculated if axes are conveniently rotated to reduce the shear stresses to zero. Figure 4, shown that idea. In the left side, there is a random state of stresses in the bidimensional plane which, for simplification, can be drawn as in the middle diagram. When the axes are rotated, due to equilibrium conditions, the state of the stress (normal and shear) change, having new values of normal and shear stress.

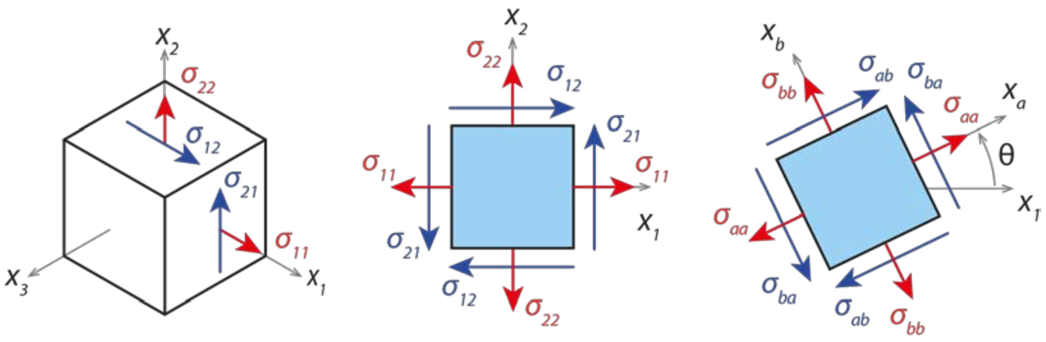

Fig. 4. At left, cube body in a bidimensional state of stress, showing the tree orthogonal axes. At center, bidimensional representation. At right, rotation around x_3 axis for calculation of new stress condition. The new oriented magnitudes are represented by letters indexes.

The equation of the stress transformation can be deducted using a section of the cubic differential body. This body contain the original condition and, in the sectioned area, the stresses in the rotated axes, which have letter indexes for differentiation from the original stress condition. For equilibrium, it is necessary to consider a free-body diagram, with forces, not stresses. Right-side of Figure 5 shown this.

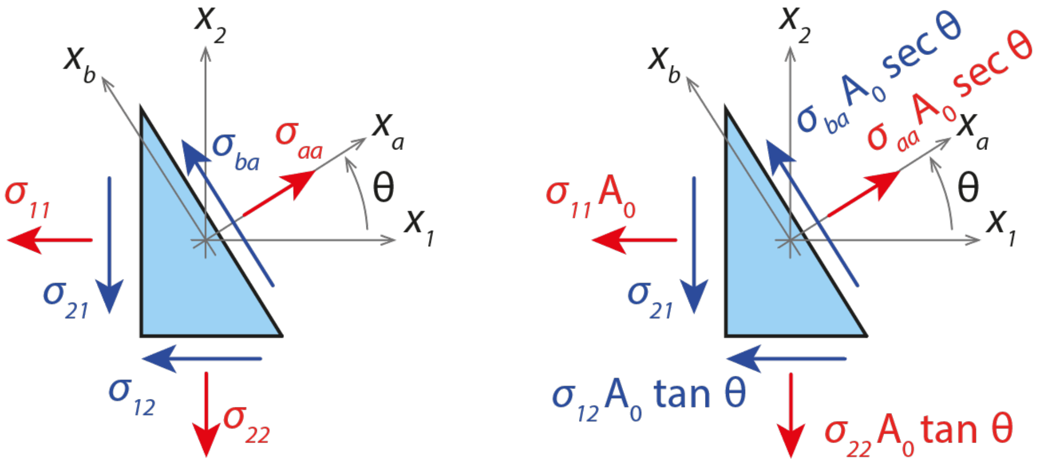

Fig. 5. At left, section of the cubic differential body with the original and rotated stress state. At right, free-body diagram, considering the areas of each of the stresses and the rotation angle.

With the rotated axes as the reference system, it is possible to apply equilibrium equation in x_a and x_b. So, considering forces in x_a,

$$\sigma_{aa}A_0 \sec\theta - \sigma_{11}A_0 \cos\theta - \sigma_{21}A_0 \sin\theta - \sigma_{22}A_0 \tan\theta \sin\theta - \sigma_{12}A_0 \tan\theta \cos\theta = 0$$

In a similar way, equilibrium conditions at x_b gives,

$$\sigma_{ba}A_0 \sec\theta + \sigma_{11}A_0 \sin\theta - \sigma_{21}A_0 \cos\theta - \sigma_{22}A_0 \tan\theta \cos\theta + \sigma_{12}A_0 \tan\theta \sin\theta = 0$$

Now, considering the symmetricity of the stress tensor ($\sigma_{12} = \sigma_{21}$) and trigonometric identities, the next equations describe the stress transformation in the bidimensional state.

$$\sigma_{aa} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta + \sigma_{21}\sin 2\theta \qquad (2.3.4)$$

$$\sigma_{ab} = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \sigma_{21} \cos 2\theta \tag{2.3.5}$$

where σ_{aa} and σ_{ab} represent the rotated stress setting, σ_{11}, σ_{22} and σ_{21} are the original stress condition and θ is the rotation angle.

Looking at equations 2.3.4 and 2.3.5 it is easy to discover that these are the parametric equations of a circle, where θ is the parameter. From here is where the Mohr's circle is born for the bidimensional case, which is a powerful graphical tool to study the stress state of any point in a body.

These equations reveal some cases from where special loads with stress states can be described. For example, considering normal stresses as zero exposes the body to simple shear condition. Applying the transformation of stress for an angle of 45 degrees, the simple shear state is transformed in pure normal stress condition (Figure 6). If the material is brittle, it must fail in a plane perpendicular to the tension state, failing at 45 degrees from a simple shear condition.

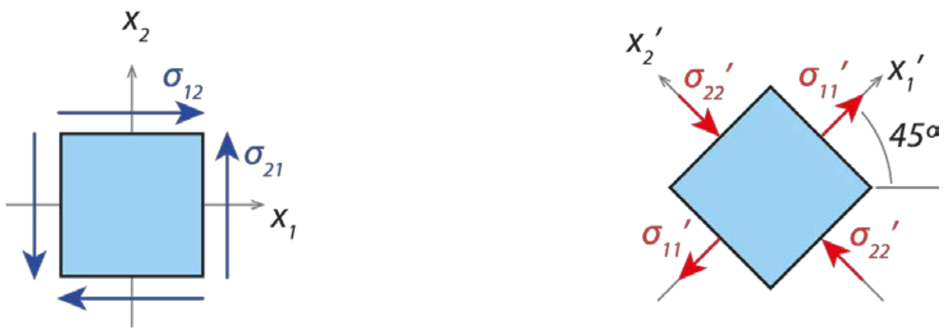

Fig. 6. Cubic body exposed to simple shear bidimensional stress (left). Using transformation of stress, it is possible to discover that simple shear produces pure normal stresses at an angle of 45 degrees. For a brittle material, it might represent failure at a plane perpendicular to the tensional stress state.

The principal stresses have importance because there are usually associated with faults, especially when the main stress is in tension, where the brittle rocks usually fail. These stresses can be calculated by deriving the last equations with respect of theta, solving for it, and substitute in the original equation.

$$\frac{d\sigma_{aa}}{d\theta} = -(\sigma_{11} - \sigma_{22}) \sin 2\theta + 2\sigma_{21} \cos 2\theta$$

From where the principal angle is calculated as

$$tan\ 2\theta_P = \frac{2\sigma_{12}}{\sigma_{22} - \sigma_{11}} \tag{2.3.6}$$

where θ_P is the principal angle of a bidimensional stress state.

By doing so, the principal stresses in bidimensional state of load can be written, after some manipulation for substituting the angle, as

$$\sigma'_{11}, \sigma'_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \tag{2.3.7}$$

where σ'_{11} and σ'_{22} are the principal stresses in the bidimensional case, as a function of the original stress state.

A similar procedure can be derived for the tridimensional state of stress. However, this load condition is not very common, and most of the real geology problems can be enough satisfied by the bidimensional cases. Anyway, in the tridimensional case, it is useful to consider the average of the principal stresses. This magnitude receives the name of pressure P, and it is calculated as:

$$P = -\frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3} \quad (2.3.8)$$

where P is the pressure and at the right side are the principal stresses. From the convention of positives, stresses are considered positive when outwards. However, positive pressure is considered inward, explaining the use of the minus sign in equation 2.3.8

3.3 Special stress states

There are many special states of stress that are created by common load conditions. Here are defined some of the more commonly states of stress that are used in the practice.

The uniaxial tension consists in only one of the principal stresses being non-zero and positive. This is commonly used for modeling fracture formation in areas exposed to extension.

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.3.9)$$

where $\sigma_{22} > 0$.

The uniaxial compression, in the other hand, is the opposite. In there, one principal stress in non-zero and negative. This model is useful in studies of faulting and folding. The stress tensor is the same as equation 2.3.9, but in this case $\sigma_{22} < 0$ (Figure 7)

The biaxial tension consists in both σ_{11} and σ_{22} nonzero and positive, and σ_{33} is zero. In engineering problems, containers with pressurized fluid are in this stress condition. In geology, as an analogy of pressure containers, this stress setting may occur at the surface of a magma chamber and in surfaces of various rock domes.

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.3.10)$$

where $\sigma_{11} > 0$ and $\sigma_{22} > 0$

Biaxial compression involves both σ_{22} and σ_{33} nonzero and negative, and σ_{11} is zero. This is a commonly assumed stress field for analyzing faulting in the crust, specially in areas of convergent plate boundaries.

$$[\sigma_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \quad (2.3.11)$$

where $\sigma_{22} < 0$ and $\sigma_{33} < 0$.

Fig. 7. Schematic diagrams for stress state and deformation of a body subjected to simple shear (left) and pure compression (right).

In the pure shear condition, $\sigma_{11} = -\sigma_{33}$ and both are nonzero, although $\sigma_{22} = 0$. In engineering, this state corresponds to tubes subjected to torsion. In geology, this is sometimes assumed for deformation in the crust, mainly in sites with connection with fault zones.

In the simple shear condition, parallel planes in a body remain parallel and maintain a constant distance, while translating relative to each other. Since the body rotates, the stress tensor is not symmetric.

$$[\sigma_{ij}] = \begin{bmatrix} 0 & \sigma_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3.12}$$

In isotropic (hydrostatic) compression, all possible planes are subjected to equal compressive stress and all the principal stresses are equal and compressive, with any shear stress present. This is the common state at great depths in the crust, taking the name of lithostatic stress.

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{lith} & 0 & 0 \\ 0 & \sigma_{lith} & 0 \\ 0 & 0 & \sigma_{lith} \end{bmatrix} \tag{2.3.12}$$

where σ_{lith} is the hydrostatic compression named, in this case, by lithostatic effects. Here, $\sigma_{\text{lith}} < 0$.

The general stress is the load condition described as a tridimensional case, where $\sigma_{11} > \sigma_{22} > \sigma_{33}$. This is the most functional state of the stress in the brittle zone of the crust.