Student ID: C1SM6027

Name: Miho Furukawa

2. Elasticity

6. Frequently used elastic constants

6.1 Bulk modulus

Bulk modulus (K) is increase in hydrostatic pressure (ΔP) by volume (V) decrease. It is defined as the amount of pressure increase with respect to the ratio of the volume decrease:

$$K \equiv \frac{\Delta P}{(-\Delta V/V)} \tag{2.6.1}$$

where K is the bulk modulus, P is the hydrostatic pressure and V is the volume. Bulk modulus has already been defined in the chapter 1. Different from that chapter, it is defined again using the Lame's constants (λ, μ) in this section. The outline for obtaining the definition is as follows: first, we express hydrostatic pressure and volume by the Lame's constants, and then, we get bulk modulus by combining them.

Since bulk modulus is defined for an isotropic stress field, no shear strain but only normal strains with the same magnitude are produced. The strains are therefore expressed as:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \tag{2.6.2}$$

$$\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0 \tag{2.6.3}$$

where ε_1, ε_2, ε_3 are normal strains and ε_4, ε_5, ε_6 are shear strains according to the Voigt notation. By substituting these values of the strains into the generalized Hooke's law for isotropic solids (the formula (2.5.31)),

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_1 \\ \varepsilon_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.6.4}$$

is given. This yields normal stresses as

$$\sigma_{1,2,3} = 3\lambda\varepsilon_1 + 2\mu\varepsilon_1 = (3\lambda + 2\mu)\varepsilon_1 \tag{2.6.5}$$

where $\sigma_{1,2,3}$ the value of normal stresses, λ and μ are the Lame's constants, and ε_1 is normal strain. With the normal stress, increase in hydrostatic pressure ΔP follows:

$$\Delta P = -\frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -(3\lambda + 2\mu)\varepsilon_1 \tag{2.6.6}$$

because the hydrostatic pressure ΔP is the averaged normal stresses.

Volume is also expressed with the normal strain. As introduced in the section 2.2, relative volume change ($\Delta V/V$) is sum of normal strains. By considering together with the formula (2.6.2),

$$\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\varepsilon_1 \tag{2.2.13}$$

is obtained. This formula is also explained by the figure 1. Assume a small cube with length L on a side. The volume of the cube (V) is

$$V = L^3$$

. If one direction is shortened by ΔL, normal strain of that direction (ε_1) is defined as:

$$\varepsilon_1 = \frac{\Delta L}{L}$$

and the volume change due to that strain is

$$\Delta L \cdot L^2$$

. Since a hydrostatic pressure is supposed, this volume change is applied to all three directions. Therefore, the total volume change (ΔV) of the cube becomes

$$3\Delta L \cdot L^2$$

and the relative volume change ($\Delta V/V$) follows:

$$\frac{\Delta V}{V} = \frac{3\Delta L \cdot L^2}{L^3} = 3\frac{\Delta L}{L} = 3\varepsilon_1$$

.

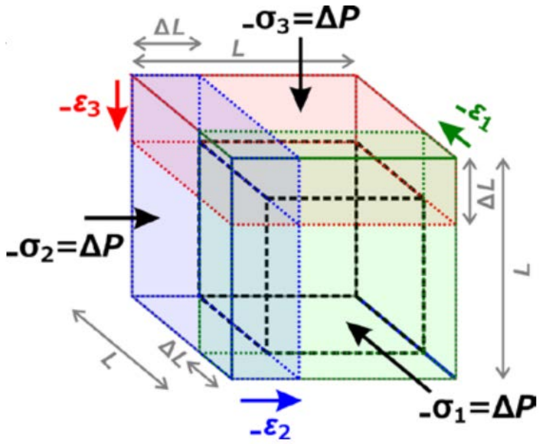

Fig. 1. A conceptional diagram of volume decrease. A small cubic with length L on a side is compressed by the hydrostatic pressure (ΔP). All three directions are shortened by ΔL. The green, blue and red cuboids denote volume decrease by the normal strain ε_1, ε_2, ε_3, respectively. The smaller cube shown with black dashed lines denotes volume after the compression. Note '-1' is multiplied by normal stress for expressing hydrostatic pressure (ΔP), since the positive direction of pressure is outward while that of stress is inward.

From the above, the bulk modulus (K) is defined:

$$K \equiv -\frac{\Delta P}{(\Delta V/V)} = -\frac{-(3\lambda + 2\mu)\varepsilon_1}{3\varepsilon_1} = \lambda + \frac{2}{3}\mu \qquad (2.6.7)$$

where K is the bulk modulus, ΔP is the hydrostatic pressure, ΔV is the volume change, V is the volume, λ and μ are the Lame's constants and ε_1 is the normal strain.

6.2 Young's modulus

In this section, we aim to define the Young's modulus (E) by the Lame's constants (λ, μ). Young's modulus (E) is the ratio of the uniaxial stress to the strain in the direction of the uniaxial stress without other stresses:

$$\sigma_1 = E\varepsilon_1 \quad \text{where } \sigma_{i\neq 1} = 0 \qquad (2.6.8)$$

where σ_1 is the uniaxial stress, E is the Young's modulus and ε_1 is the strain. This stress field is shown in the figure 2. In the figure 2, uniaxial stress (σ_1) is applied to a cube. This produces shrinkage in the direction of ε_1 and expansion in the directions of ε_2 and ε_3. We simply compare the strain ε_1 to the uniaxial stress (σ_1) when we define the Young's modulus. However, strains parallel (ε_1) and vertical

(ε_2, ε_3) to the uniaxial stress are mathematically related. Using these relations, we derive the Young's modulus using the Lame's constants.

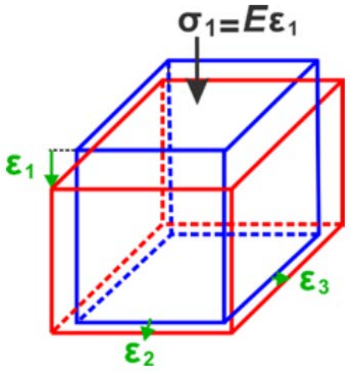

Fig. 2. A conceptual diagram of the uniaxial stress (σ_1) and the three normal strains ($\varepsilon_{1,2,3}$). When the cube shown with the blue lines is compressed by the stress, it shrinks in the direction of ε_1 while expands in the directions of ε_2 and ε_3, turning into the cuboid shown with the red lines.

We rewrite the formula (2.6.8) into the form of the generalized Hooke's law for isotropic solids. From the formula (2.6.8),

$$\sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$$

is given, and by substituting it into the formula (2.5.31),

$$\begin{bmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix} \quad (2.6.9)$$

is obtained, where σ_1 is the uniaxial stress, λ and μ are the Lame's constants and $\varepsilon_{1,2,3,4,5,6}$ are the strains. The formula (2.6.9) denotes the relations of the Young's modulus with the Lame's constants in the case of isotropic materials. From the equation (2.6.9), we have

$$\sigma_1 = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 \quad (2.6.10)$$

$$\sigma_2 = 0 = \lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3 \quad (2.6.11)$$

$$\sigma_3 = 0 = \lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2\mu)\varepsilon_3 \quad (2.6.12)$$

. Therefore, by combining the equation (2.6.8) $\sigma_1 = E\varepsilon_1$and (2.6.10), we have an equation between the three strains ($\varepsilon_{1,2,3}$):

$$E\varepsilon_1 = (\lambda + 2\mu)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$$

, which is equivalent to

$$(\lambda + 2\mu - E)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3 = 0 \quad (2.6.13)$$

. From the three equations above ((2.6.11), (2.6.12), (2.6.13)), we erase the three strains (ε_1, ε_2, ε_3). After erasing them, we have the formula expressing the Young's modulus (E) using the Lame's constants (λ, μ). By combining the equation (2.6.11) and (2.6.13), we have

$$\lambda\varepsilon_1 + (\lambda + 2\mu)\varepsilon_2 + \lambda\varepsilon_3 = (\lambda + 2\mu - E)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$$

, which is equivalent to

$$\varepsilon_2 = \frac{2\mu - E}{2\mu}\varepsilon_1 \tag{2.6.14}$$

. Similarly, by combining the equation (2.6.12) and (2.6.13), we have

$$\lambda\varepsilon_1 + \lambda\varepsilon_2 + (\lambda + 2\mu)\varepsilon_3 = (\lambda + 2\mu - E)\varepsilon_1 + \lambda\varepsilon_2 + \lambda\varepsilon_3$$

, which is equivalent to

$$\varepsilon_3 = \frac{2\mu - E}{2\mu}\varepsilon_1 \tag{2.6.15}$$

. By substituting the equation (2.6.14) and (2.6.15) into the equation (2.6.13), we have

$$(\lambda + 2\mu - E)\varepsilon_1 + \lambda\frac{2\mu - E}{2\mu}\varepsilon_1 + \lambda\frac{2\mu - E}{2\mu}\varepsilon_1 = 0$$

and therefore,

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{2.6.16}$$

by simplification. The formula (2.6.16) shows the relation between the Young's modulus (*E*) and the Lame's constants (λ, μ). This means that the Young's modulus (*E*) can be expresses using the bulk modulus (*K*) and rigidity G=μ, because the bulk modulus (*K*) is expressed by the Lame's constants (λ, μ) as the formula (2.6.7) denotes. From the formula (2.6.7), we have

$$\lambda = K - \frac{2}{3}\mu = K - \frac{2}{3}G$$

. By substituting the equation into the equation (2.6.16),

$$E = \frac{9KG}{(3K + G)} \tag{2.6.17}$$

is obtained, which gives the relation between the Young's modulus (*E*), the bulk modulus (*K*) and rigidity (*G*).

6.3 Poisson's ratio

When a uniaxial stress is applied, the dimension of the body normal to the uniaxial stress increases, as shown in figure 3. Poisson's ratio (ν) is defined as the negative ratio of the transverse strain to the axial strain:

$$\nu \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{\varepsilon_3}{\varepsilon_1} \tag{2.6.17}$$

where $\sigma_1 = E\varepsilon_1$, and $\sigma_{2,3}=0$. (The formula should be numbered as (2.6.18) because (2.6.17) has already used. However, to prevent confusion, the numbers are not corrected below.)

The Poisson's ratio (ν) can be expressed using the other constants introduced above. By substituting the equation (2.6.14) into the equation (2.6.17),

$$\nu = -\frac{\frac{\varepsilon_1(2\mu - E)}{2\mu}}{\varepsilon_1} = -\frac{2\mu - E}{2\mu} \tag{2.6.18}$$

is obtained, which means that the Poisson's ratio (ν) can be expressed using the Young's modulus (*E*) and rigidity (μ). (In the lecture material, the minus sign in the right-hand side is missing. Here, the minus sign is added as a correction.) By substituting the formula (2.6.16) into the formula (2.6.18), the Poisson's ratio (ν) can be expressed by the Lame's constants (λ, μ):

$$\nu = -\frac{2\mu - \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}}{2\mu} = \frac{\lambda}{2\lambda + 2\mu} \tag{2.6.19}$$

. Similarly, by combining the equation (2.6.19) with the equation (2.6.7) and $G=\mu$,

$$\nu = \frac{K - \frac{2}{3}G}{2\left(K - \frac{2}{3}G\right) + 2G} = \frac{3K - 2G}{6K + 2G} \tag{2.6.20}$$

, where the Poisson's ratio (ν) can be expressed using the bulk modulus (K) and rigidity (G).

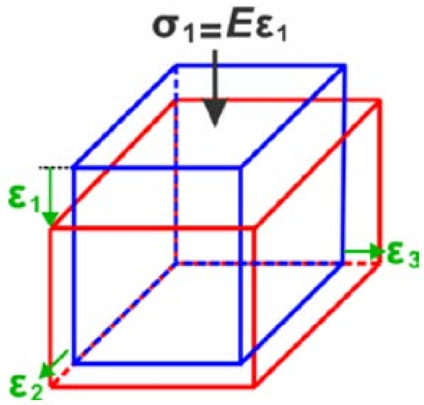

Fig. 3. A conceptual diagram of the uniaxial stress (σ_1) and the three normal strains ($\varepsilon_{1,2,3}$). When the cube shown with the blue lines is compressed by the stress, it shrinks in the direction of ε_1 while expands in the directions of ε_2 and ε_3, turning into the cuboid shown with the red lines. Different from the figure 2, one of the corners of the blue cube and red cuboid are aligned so that the arrows of the strains indicate the actual lengths of the deformation.

The Poisson's ratio (ν) has upper and lower limits, which are derived by the equation (2.619) and (2.6.20). The equation (2.6.20) is modified as:

$$\nu = \frac{3K - 2G}{6K + 2G} = \frac{K - \frac{2}{3}G}{2\left(K + \frac{1}{3}G\right)}$$

, which shows that

$$\nu \sim \frac{-\frac{2}{3}G}{\frac{2}{3}G} = -1 \quad when\ G \gg K$$

. From the equation (2.6.19),

$$\nu = \frac{\lambda}{2\lambda + 2\mu} \sim \frac{\lambda}{2\lambda} = 0.5 \quad when\ \lambda \gg \mu$$

. As a side note, this $\nu \rightarrow 0.5$ can be derived also from the equation (2.6.20). When $G \ll K$,

$$\nu \sim \frac{K}{2K} = 0.5$$

is obtained. (This is an additional information to the lecture material.) To summarize above, we have

$$-1 < \nu < +0.5 \tag{2.6.21}$$

as the possible range of the Poisson's ratio (v).

6.4 Examples of several elastic constants

In this section, we see some examples of bulk modulus (K) and rigidity (G) of some solid materials. The materials are not isotropic but crystalline solids, so discussion here is made by assumpting the elasticity of crystalline solids as isotropic solids. We compare ten kinds of materials: corundum (Al2O3), periclase (MgO), calcium oxide (CaO), forsterite (Mg2SiO4), fayalite (Fe2SiO4), intermediate component of forsterite and fayalite ((Mg0.9Fe0.1)2SiO4), spinel (MgAl2O4), pyrope garnet (Mg3Al2Si3O12), potassium chloride (KCl) and sodium chloride (NaCl).

First, we see bulk modulus (K) against temperature (T) at ambient pressure (figure 4). The bulk modulus (K) indicates how the solid is difficult to compress. As the figure 4 shows, bulk modulus (K) decreases only slightly with temperature increase. Moreover, the decreasing rate of bulk modulus (K) against temperature increase is almost constant because the lines are almost linear. Corundum (Al2O3), which is one of the most incompressible solids, has the highest bulk modulus (K) reaching 250 GPa, and the second highest is spinel (MgAl2O4). Following them is pyrope garnet (Mg3Al2Si3O12), periclase (MgO) and olivine minerals (Fe2SiO4, Mg2SiO4, (Mg0.9Fe0.1)2SiO4) in the order. Within olivine minerals, fayalite (Fe2SiO4) is higher than forsterite (Mg2SiO4), and the intermediate component ((Mg0.9Fe0.1)2SiO4) and forsterite (Mg2SiO4) are almost the same. Calcium oxide (CaO) is slightly lower than the olivine minerals. Both periclase (MgO) and calcium oxide (CaO) have the rock salt structure, but the former is higher than the olivine minerals and the latter is lower than those. Alkali halide is distinctively low bulk modulus (K) than the oxide, especially potassium chloride (KCl) which is heavier than sodium chloride (NaCl). From the above, we conclude that oxide has the higher bulk modulus (K) than alkali halide, aluminum (Al) and magnesium (Mg) minerals have high bulk modulus (K), and Fe-Mg substitution does not affect the bulk modulus (K) significantly.

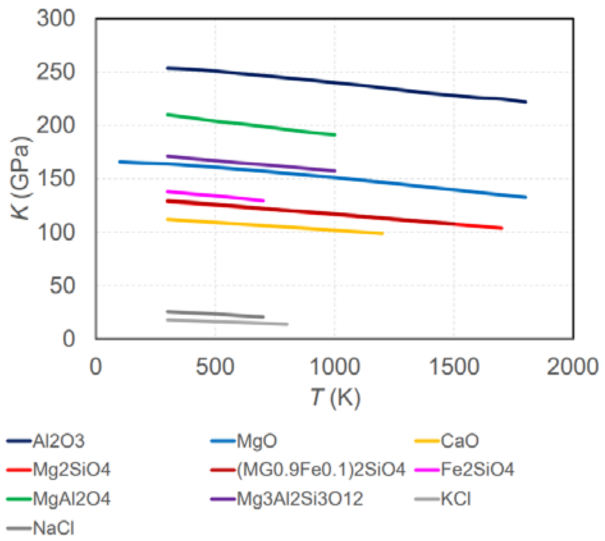

Fig. 4. A graph of bulk modulus (K) of several materials against temperature (T) at ambient pressure. All the materials shown here are crystalline solids, so we assumpt the elasticity of the crystalline solids as isotropic solids. The color of each line corresponds with the caption below the graph. See text for detailed explanation.

Next, we see rigidity (G) against temperature at ambient pressure (figure 5). Rigidity (G) indicates how difficult to shear the solids elastically. The order from high to low rigidity (G) is similar to that of the bulk modulus (K): corundum (Al2O3), periclase (MgO) and spinel (MgAl2O4) have high rigidity

(G), which means that they are difficult to shear. Strictly speaking, the order of spinel ($MgAl_2O_4$) and periclase (MgO) is opposite between bulk modulus (K) and rigidity (G). The next high rigidity (G) is pyrope garnet ($Mg_3Al_2Si_3O_{12}$), then we have olivine minerals (Mg_2SiO_4, $(Mg_{0.9}Fe_{0.1})_2SiO_4$, Fe_2SiO_4). A difference of the rigidity from the bulk modulus (K) is Fe end member of olivine, fayalite (Fe_2SiO_4), has much lower rigidity (G) than Mg rich olivine minerals (Mg_2SiO_4 and $(Mg_{0.9}Fe_{0.1})_2SiO_4$). In the case of the bulk modulus (K), fayalite (Fe_2SiO_4) was higher than forsterite (Mg_2SiO_4). Therefore, the effect of Fe-Mg substitution is opposite and more prominent in the rigidity (G) compared with in the bulk modulus (K). Alkali halide solids (potassium chloride (KCl) and sodium chloride (NaCl)) have much lower rigidity (G) than oxide, which is the same as the bulk modulus (K).

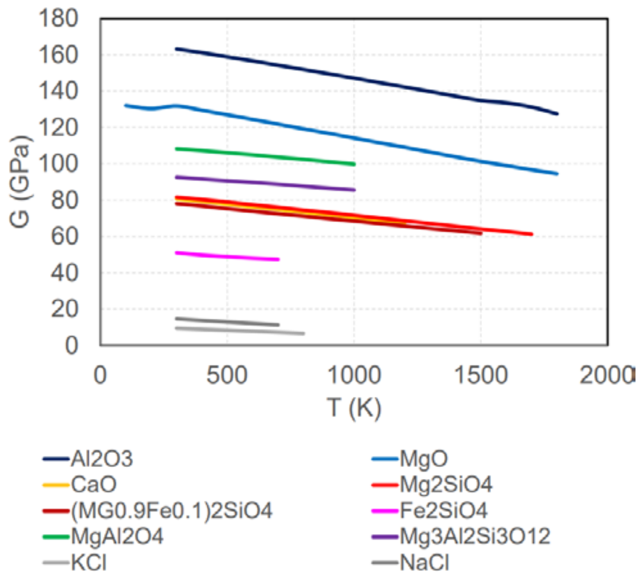

Fig. 5. A graph of rigidity (G) of several materials against temperature (T) at ambient pressure. All the materials shown here are crystalline solids, so we assumpt the elasticity of the crystalline solids as isotropic solids. The color of each line corresponds with the caption below the graph. See text for detailed explanation.

Both the bulk modulus (K) and rigidity (G) decrease with temperature increase, so we compare the dependence of the two elastic constants (K (K_s in the figure) and G) on the temperature increase in figure 6. The left diagram shows the bulk modulus (K) at high temperatures normalized by the bulk modulus of each solid at ambient temperature. The right diagram shows the rigidity (G) at high temperatures normalized by the rigidity of each solid at ambient temperature. The ambient temperature is defined as 300 K for both the bulk modulus (K) and rigidity (G). Thus, the diagrams show the relative temperature dependence of the bulk modulus (K) and rigidity (G) for the ten materials. By the definition, the values at the temperature of 300 K are unity in these figures. As the figure 6 shows, the rigidity (G) has steeper slopes than the bulk modulus (K). Therefore, the rigidity (G) decreases more rapidly with temperature increase than the bulk modulus (K): in other words, the rigidity (G) has a stronger dependence on temperature.

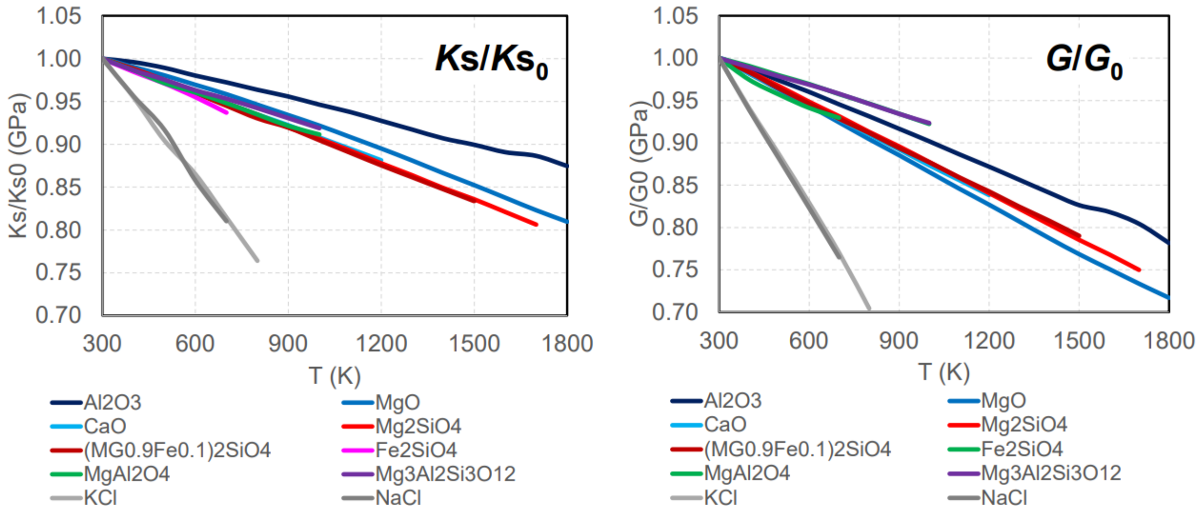

Fig. 6. Graphs of normalized bulk modulus (*K* (*K*s in the figure)) and rigidity (*G*) against temperature (*T*). Left: The bulk modulus (*K*s) normalized by the bulk modulus at ambient temperature of a 300 K (Ks_0). Right: The rigidity (*G*) normalized by the rigidity at ambient temperature of 300 K (G_0). The materials shown here are crystalline solids, so we assumpt the elasticity of the crystalline solids as isotropic solids. The color of each line corresponds with the captions below the graphs. See text for detailed explanation. (Currently, both Fe2SiO4 and MgAl2O4 are shown in green in the right graph, but considering from the left graph, the caption and line for Fe2SiO4 in the right graph may have been supposed to be in pink.)