Mineral Physics I
Chapter 5. Heat transfer properties
Section 3. Radiative thermal conductivity

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This section

- Basic laws of thermal radiation
  - Planck’s law of thermal radiation
  - Stephan-Boltzmann’s law
- Radiation laws in a transparent medium due to the slower light velocity
- Thermal resistance of radiative heat transfer: optical absorption
  - Photon mean free path

Thermal radiation

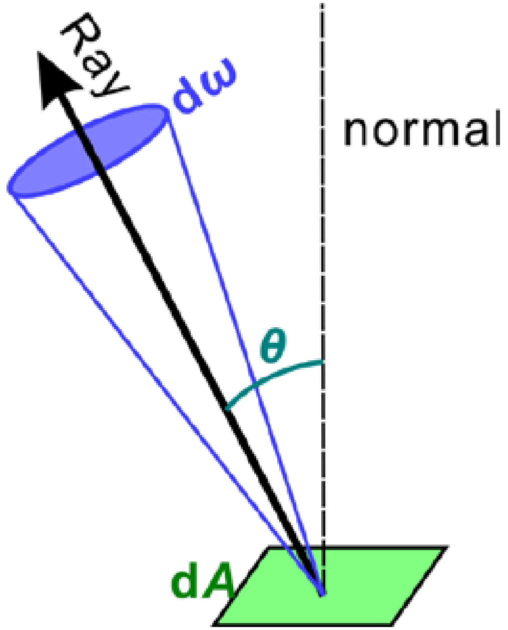
- A hot body emits electromagnetic waves: thermal radiation
  - Electromagnetic wave generated by the thermal motion of charged particles in matter
    - All matter with a *T* greater than 0 K emits
  - Internal energy of lattice vibration is converted to electromagnetic waves to be released to space
- Basic features of thermal radiation
  - Becoming stronger with increasing *T*
  - Color changes from red to white with increasing *T*
    - Continuous spectra

Terms

- Radiance
  - the radiant energy flux from a surface per unit solid angle and per unit area of radiating surface
- Radiant emittance: the radiant energy flux emitted by a surface per unit area
- Spectral radiance
  - Radiance at a given frequency or wavelength
- Spectral energy density
  - Stored energy per unit volume at a given frequency or wavelength

Specific intensity

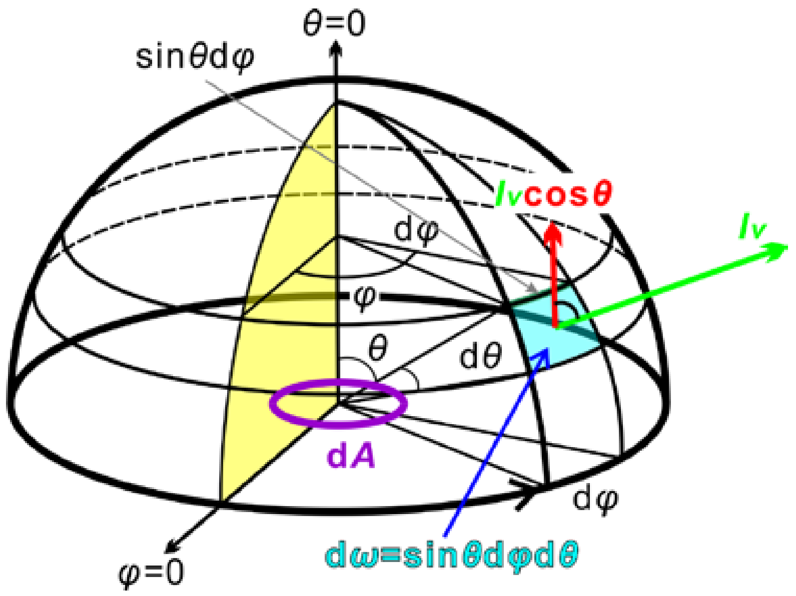
- Measures of energy flow: specific Intensity and Flux
  - The amount of energy $\mathrm{d}E_\nu$ transported through a surface area $\mathrm{d}A$ is proportional to $\mathrm{d}t$ (length time), $\mathrm{d}\nu$ (frequency width), $\mathrm{d}\omega$ (solid angle) and the projected unit surface area $\cos\theta\ \mathrm{d}A$
  - The proportionality factor: the specific intensity $I_\nu(\cos\theta)$ (J m^{-2} sr^{-1} Hz^{-1} s^{-1})
    - a function of the direction to the normal
  - $\mathrm{d}E_\nu = I_\nu(\cos\theta)\cos\theta\ \mathrm{d}A\mathrm{d}\omega\mathrm{d}\nu\mathrm{d}t$ (5.3.1)
  - $I_\lambda = \frac{c}{\lambda^2} I_\nu$ (5.3.2)
    - $I_\lambda \mathrm{d}\lambda = I_\nu \mathrm{d}\nu,\ \nu = \frac{c}{\lambda}$ → $\mathrm{d}\nu = -\frac{c}{\lambda^2}\mathrm{d}\lambda$



Radiative flux

- q Energy flows through a surface element d*A*
  - Ø $dE_\nu \propto I_\nu \cos\theta \, \mathrm{d}\omega$
- q Flux, πF_ν: integration of the specific intensity over the half solid angle
  - Ø Upper side
  - Ø $\Omega = 2\pi$
  - Ø $F_\nu = \int_{2\pi} I_\nu \cos\theta \, \mathrm{d}\omega$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} I_\nu \cos\theta \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi \quad (5.3.3)$$

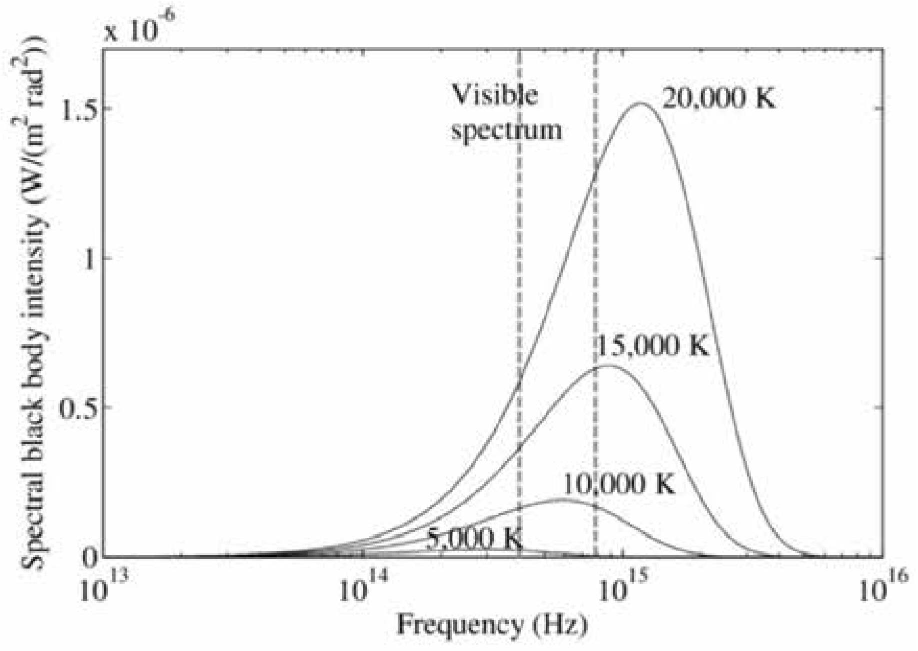


Planck's law of thermal radiation

- Planck's law: the spectral density of electromagnetic radiation emitted by a black body in thermal equilibrium at a given temperature *T*

  - $B(\nu, T)\mathrm{d}\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T)-1}\mathrm{d}\nu$ (5.3.4)

    - In the freq. range between ν and $\nu + \mathrm{d}\nu$
      - h: Planck's constant
      - c: is the speed of light
      - k_B: Boltzmann's constant

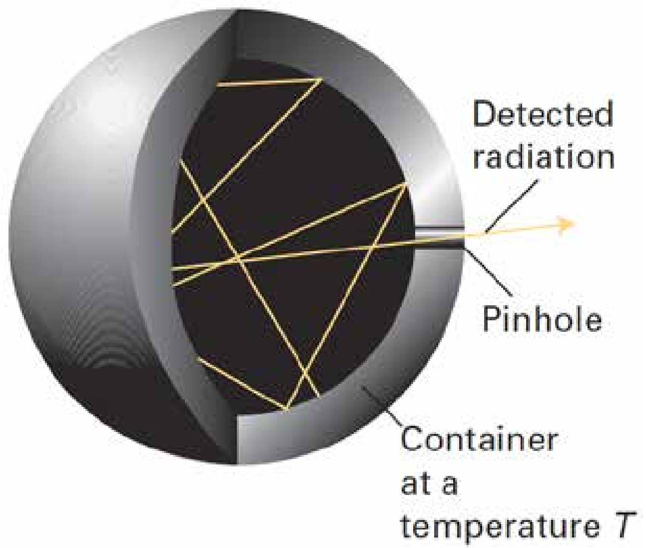


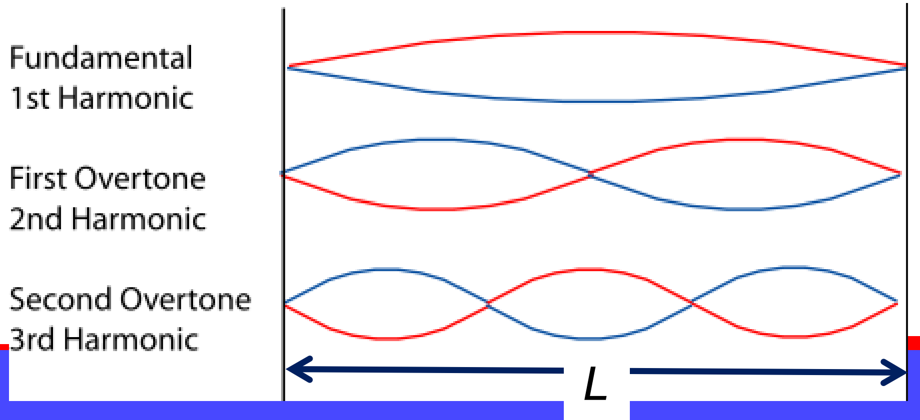
Planck's law of thermal radiation

- Spectral radiance form (J m^{-2} sr^{-1} s^{-1})
  - With frequency: $B_\nu(\nu, T)\mathrm{d}\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T)-1}\mathrm{d}\nu$ (5.3.4)
  - With wavelength: $B_\lambda(\lambda, T)\mathrm{d}\nu = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k_B T)-1}\mathrm{d}\nu$ (5.3.5)
- Spectral energy density form (J m^{-3})
  - $u(T) = \frac{4\pi}{c}B(T)$ (5.3.6)
  - With frequency: $u_\nu(\nu, T)\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/k_B T)-1}\mathrm{d}\nu$ (5.3.7)
  - With wavelength: $u_\lambda(\lambda, T)\mathrm{d}\nu = \frac{8\pi hc}{\lambda^5}\frac{1}{\exp(hc/\lambda k_B T)-1}\mathrm{d}\nu$ (5.3.8)

Derivation of Planck's law -1
EM-wave in a blackbody container

- A black body: an object capable of emitting and absorbing all wavelengths of light completely
- An empty black-body container maintained at a constant temperature
  - Any radiation: thermal equilibrium with the wall by absorption and (re-)emission
  - Dimension of the cavity: L
    - The stationary wave
      - $\lambda = 2L, \frac{2L}{2}, \frac{2L}{3}, \ldots, \frac{2L}{n}, \ldots$ (5.3.9)
      - $\nu = \frac{c}{\lambda_n} = \frac{nc}{2L}$ (5.3.10)
- Leaking-out of the light through a pinhole





Derivation of Planck's law -2
Stationary EM-wave

- A set of stationary waves of thermal radiation
  - $E_x = E_{x0} \sin \omega t \cos(n_x \pi x/L) \sin(n_y \pi y/L) \sin(n_z \pi z/L)$ (5.3.11a)
  - $E_y = E_{y0} \sin \omega t \sin(n_x \pi x/L) \cos(n_y \pi y/L) \sin(n_z \pi z/L)$ (5.3.11b)
  - $E_z = E_{z0} \sin \omega t \sin(n_x \pi x/L) \sin(n_y \pi y/L) \cos(n_z \pi z/L)$ (5.3.11c)
    - E_x, E_y, E_z: electric fields in the x, y, and z direction

- Substituting (5.3.11a)-(5.3.11c) into the wave equation: $\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = c^2 \nabla^2 \boldsymbol{E}$
  - $\omega^2 \boldsymbol{E} = c^2 \frac{(n_x^2+n_y^2+n_z^2)\pi^2}{L^2} \boldsymbol{E}$ → $\omega^2 = \frac{\pi^2 c^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$ (5.3.12)
  - $\nu = \frac{c}{2L}(n_x^2 + n_y^2 + n_z^2)^{\frac{1}{2}}$

- Define $n^2 \equiv n_x^2 + n_y^2 + n_z^2$ (5.3.13)
  - $\nu = \frac{c}{2L}(n_x^2 + n_y^2 + n_z^2)^{\frac{1}{2}} = \frac{c}{2L} n$ (5.3.14)

Derivation of Planck's law -3
Density of modes

- The density of modes between n and $n + \mathrm{d}n$, $D(n)\mathrm{d}n$
  - Directions of vibration (polarization) of $\boldsymbol{E}$: 2
  - $n = \left(n_x^2 + n_y^2 + n_z^2\right)^{\frac{1}{2}}$ → spherical distribution in an n-space:
    - $4\pi n^2 \mathrm{d}n$ modes between n and $n + \mathrm{d}n$
  - The stationary wave: $n_x, n_y, n_z > 0$
    - only $\left(\frac{1}{2}\right)^3$ of $4\pi n^2$ sphere

➢ $D(n)\mathrm{d}n = 2 \times \frac{1}{8} \times 4\pi n^2 \mathrm{d}n = \pi n^2 \mathrm{d}n$ (5.3.15)

- The density of states of photons between ν and $\nu + \mathrm{d}\nu$, $g(\nu)\mathrm{d}\nu$

➢ $g(\nu)\mathrm{d}\nu = \pi \left(\frac{2L}{c}\nu\right)^2 \frac{2L}{c}\mathrm{d}\nu = \frac{8\pi L^3}{c^3}\nu^2 \mathrm{d}\nu$ (5.3.16)

  - (5.3.9): $\nu = \frac{c}{2L}n$ → $n = \frac{2L}{c}\nu$

Derivation of Planck's law -4
Spectral energy density

- From the Planck distribution, the average energy at a given frequency, ν, at T is given by Eq. (3.5.7)
  - $\langle \varepsilon(\nu) \rangle = \frac{h\nu}{\exp(h\nu/k_B T)-1}$ (3.5.7)
- By combining (3.5.7) and (5.3.12) $g(\nu)\mathrm{d}\nu = \frac{8\pi L^3}{c^3}\nu^2 \mathrm{d}\nu$, we have the spectral *E* in the volume L^3 at a given ν
  - $U_\nu(\nu, T)\mathrm{d}\nu = \frac{h\nu}{\exp(h\nu/k_B T)-1} \frac{8\pi L^3}{c^3}\nu^2 \mathrm{d}\nu = \frac{8\pi L^3 h\nu^3}{c^3} \frac{1}{\exp(h\nu/k_B T)-1} \mathrm{d}\nu$ (5.3.17)
- The spectral energy density is obtained by normalizing (5.3.17) by $V = L^3$
  - $u_\nu(\nu, T)\mathrm{d}\nu = \frac{U(\nu,T)\mathrm{d}\nu}{L^3} = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(h\nu/k_B T)-1} \mathrm{d}\nu$ (5.3.7)
    - Planck's law of radiation thermal radiation by spectral energy density

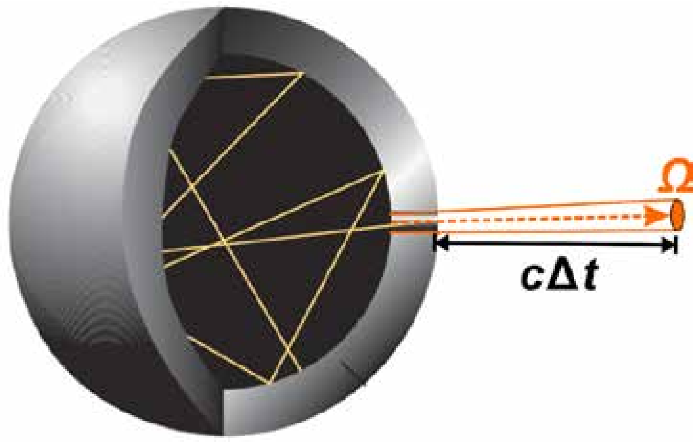
Derivation of Planck's law -5
Spectral radiance

- The spectral radiation radiance: the energy emitted per unit solid angle and per time
  - Unit direction : normalized by the whole solid angle 4π
  - Unit time : the length that the light has swept for a unit time, $c\Delta t$

- (Spectral radiance) = (Spectral energy density)*$c/4\pi$
  - $B(\nu,T)\mathrm{d}\nu = \frac{c}{4\pi}u_\nu(\nu,T)\mathrm{d}\nu = \frac{c}{4\pi}\frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/k_BT)-1}\mathrm{d}\nu$

    $$= \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_BT)-1}\mathrm{d}\nu \qquad (5.3.4)$$
  - Planck's law of radiation thermal radiation by spectral radiance

Emission leaking out from a small pinhole



Stefan–Boltzmann law

- Stefan–Boltzmann law : the total E radiated per unit surface area of a black body across all λ per unit t is directly proportional to T^4
  - Radiance: $L = \frac{\sigma}{\pi} T^4$ (5.3.18)
  - Radiant emittance: $j^* = \sigma T^4$ (5.3.19)
  - Energy density: $U = \frac{4\sigma}{c} T^4$ (5.3.20)
    - $\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3} = 5.67\ldots \times 10^{-8}$ $Wm^{-2}K^{-4}$ (5.3.21)
      - Stefan–Boltzmann constant

Derivation of Stefan–Boltzmann law -1
By radiance

- Stefan-Boltzmann law: the total energy, U, radiated from a black body is proportional to T^4.
  - Integration of Eq. (5.3.3) $B_\nu(\nu,T)\mathrm{d}\nu = \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right)-1}\mathrm{d}\nu$ over all ν

    - $L = \int_0^\infty B_\nu(\nu,T)\mathrm{d}\nu = \int_0^\infty \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right)-1}\mathrm{d}\nu = \frac{2h}{c^2}\left(\frac{k_B T}{h}\right)^4\int_0^\infty \frac{\left(\frac{h\nu}{k_B T}\right)^3}{e^{\frac{h\nu}{k_B T}}-1}\frac{h}{k_B T}\mathrm{d}\nu$
    - $L = \frac{2h}{c^2}\left(\frac{k_B T}{h}\right)^4\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x \qquad x = \frac{h\nu}{k_B T}$ (5.3.22)
    - $L = \frac{2h}{c^2}\left(\frac{k_B T}{h}\right)^4\frac{\pi^4}{15} = \frac{2\pi^4 k_B^4}{15c^2h^3}T^4 = \frac{\sigma}{\pi}T^4$ (5.3.18)
      - $\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x = \frac{\pi^4}{15}$ (5.3.23)

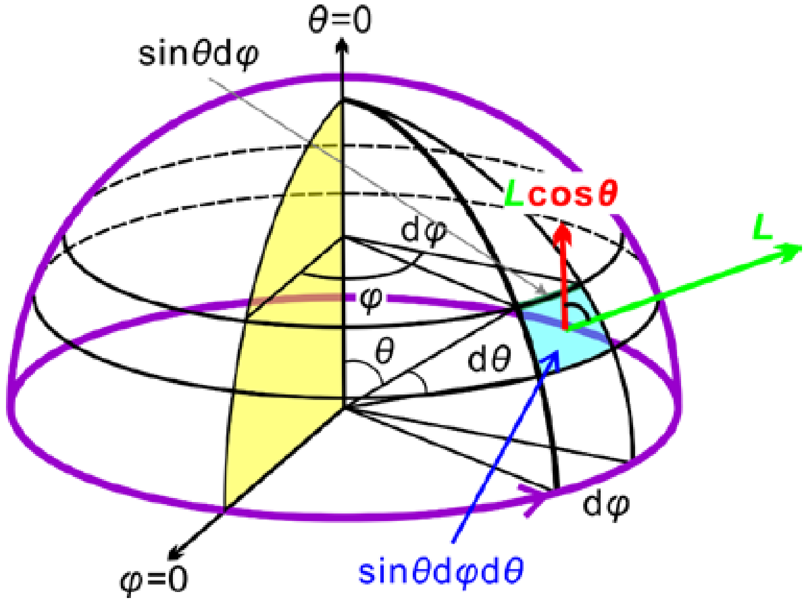
Derivation of Stefan–Boltzmann law -2

- Radiant emittance: integration of radiance L over all radiation in the direction normal to the surface
  - Radiant emittance: per area
  - Radiance: per area and per solid angle
- Normal direction: $\theta = 0$
- Radiation in the normal direction: $L \cos\theta$
- Solid angle: $\sin\theta \, d\phi d\theta$
- $j^* = \int_0^{\pi/2} \int_0^{2\pi} L \cos\theta \sin\theta \, d\phi d\theta$

$$= \frac{\sigma}{\pi} T^4 \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\sigma}{\pi} T^4 \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} d\theta \, 2\pi = \sigma \left[\frac{1}{2} \cos 2\theta\right]_0^{\frac{\pi}{2}} T^4 = \sigma T^4$$

(5.3.19)



Radiation laws in a transparent medium

q The velocity of light in a medium is slower than in vacuum: $q < c$

Ø The refractive index: $n = c/q$ (5.3.24)

q Planck's law: (5.3.4) $B(\nu, T)\mathrm{d}\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T)-1}\mathrm{d}\nu$

Ø $B_\nu^{\mathrm{med}}(\nu, T) = \frac{2h}{q^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right)-1} = \left(\frac{c}{q}\right)^2 \frac{2h}{c^2}\frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right)-1} = n^2 B_\nu(\nu, T)$ (5.3.25)

q Stefan-Boltzmann law: (5.3.18) $L = \frac{\sigma}{\pi}T^4$

Ø $L = \int_0^\infty n^2 B_\nu^{\mathrm{med}}(\nu, T)\mathrm{d}\nu = \langle n^2 \rangle \frac{\sigma}{\pi}T^4$ (5.3.26)

ü $\langle n^2 \rangle = \frac{\int_0^\infty n^2 B_\nu(\nu,T)\mathrm{d}\nu}{\int_0^\infty B_\nu(\nu,T)\mathrm{d}\nu}$ (5.3.27)

q The radiated energy in a medium is higher than vacuum, proportionally to $\langle n^2 \rangle$

Thermal resistance of radiative heat transfer

- Perfectly transparent material
  - Thermal radiation should go through the body without any interaction
    - The body is not heated
    - $q \propto \Delta T$, not $\mathrm{d}T/\mathrm{d}x$
  - No thermal conductivity by thermal radiation

- If the matter absorbs light to some degree, energy is transferred from one part of the body to another.
    - The part at higher temperature emits more light (thermal radiation)
    - The temperature of this part decreases by losing energy, whereas that of the neighboring part increases by absorbing the emitted light

- "Optical absorption" of light: essential for the radiative heat transfer

Absorbance & absorption coefficient

- The intensity of light passing through a semi-transparent body: exponentially attenuate
  - $I_{\mathrm{tr}}(s) = I_0 \exp(-\alpha s) \approx I_0(1 - \alpha s)$ (5.3.28)
    - I_{tr}: intensity of transmitted light,
    - I_0: intensity of incident light
    - α: absorption coefficient [m^{-1}]
    - s: path
  - $\mathrm{d}I = I\mathrm{d}(\exp(-\alpha s)) \approx I\mathrm{d}(1 - \alpha s) = -\alpha I \mathrm{d}s$ (5.3.29)
- Absorbance, A: the logarithm of the ratio of the incident to transmitted intensity
  - $A = \ln\left(\frac{I_o}{I_{\mathrm{tr}}}\right) = \ln\left(\frac{I_o}{I_0 \exp(-\alpha s)}\right) = \alpha s$ (5.3.30)
    - In many cases, not the natural logarithm (ln) but common logarithm ($\log_{10}$)

Absorptivity

- The intensity of the light absorbed by the body, I_{abs}
  - $I_{abs} = I_0 - I_{tr} = I_0 \exp(-\alpha d) = I_0(1 - \exp(-\alpha s))$ (5.3.31)
- Absorptivity, ξ: the fraction of total incident radiation which is absorbed by a body
  - $\xi = \frac{I_{abs}}{I_0} = \frac{I_0 - I_{\mathrm{tr}}}{I_0}$ (5.3.32)
  - $\xi = \frac{I_0 - I_0 \exp(-\alpha d)}{I_0} = 1 - \exp(-\alpha s)$ (5.3.33)

α: absorption coefficient

s: path

Photon mean free path

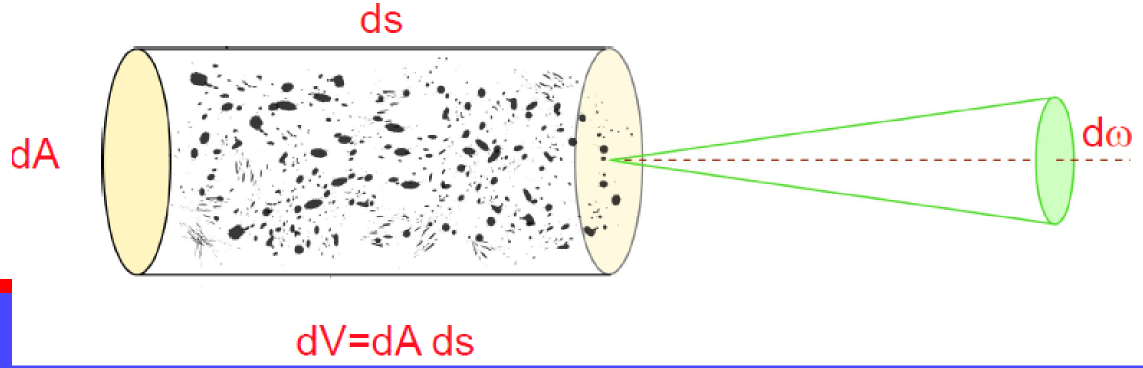
- Photon mean free path, Λ_{pht}: the average length of transmission of the light
  - The number of photon ∝ the light intensity
  - The intensity of light transmitted over s : (5.3.28) $I_{\mathrm{tr}}(s) = I_0\exp(-\alpha s)$
  - $$\Lambda_{\mathrm{pht}} = \frac{\int_0^\infty sI_0\exp(-\alpha s)\mathrm{d}s}{\int_0^\infty I_0\exp(-\alpha s)\mathrm{d}s} = \frac{\frac{1}{\alpha^2}\int_0^\infty(-\alpha)x\exp(-\alpha s)(-\alpha)\mathrm{d}s}{-\frac{1}{\alpha}\int_0^\infty \exp(-\alpha x)(-\alpha)\mathrm{d}s} = -\frac{1}{\alpha}\frac{\int_0^\infty ze^{-z}\mathrm{d}z}{\int_0^\infty e^{-z}\mathrm{d}z} = \frac{1}{\alpha} \quad (5.3.34)$$
    - $\int_0^\infty e^{-z}\mathrm{d}z = [-e^{-z}]_0^\infty = -(-e^{-\infty} - e^0) - 1$
    - $\int_0^\infty ze^{-z}\mathrm{d}z = \int_0^\infty z(-e^{-z})'\mathrm{d}z = [x(-e^{-z})]_0^\infty - \int_0^\infty (z)'(-e^{-z})\mathrm{d}z$
      $= 0 - \int_0^\infty (-e^{-z})\mathrm{d}z = 1$
  - The photon mean free path: the reciprocal absorption coefficient
    - $I_{\mathrm{tr}}(s) = I_0\exp(-s/\Lambda_{\mathrm{pht}})$ (5.3.35)

The equation of radiative heat transfer

- Energy added by emission within a volume $\mathrm{d}V = \mathrm{d}A\mathrm{d}s$
  - $\mathrm{d}E_\nu^{\mathrm{em}} = \epsilon_\nu \mathrm{d}V\mathrm{d}\omega\mathrm{d}\nu\mathrm{d}t = \epsilon_\nu \mathrm{d}A\mathrm{d}s\mathrm{d}\omega\mathrm{d}\nu\mathrm{d}t$ (5.3.36)
    - ϵ_ν: emission coefficient [J m^{-3} sr^{-1} Hz^{-1} s^{-1}]
    - $\epsilon_\nu = \varepsilon_\nu B_\nu$, ε_ν: emissivity [m^{-1}]

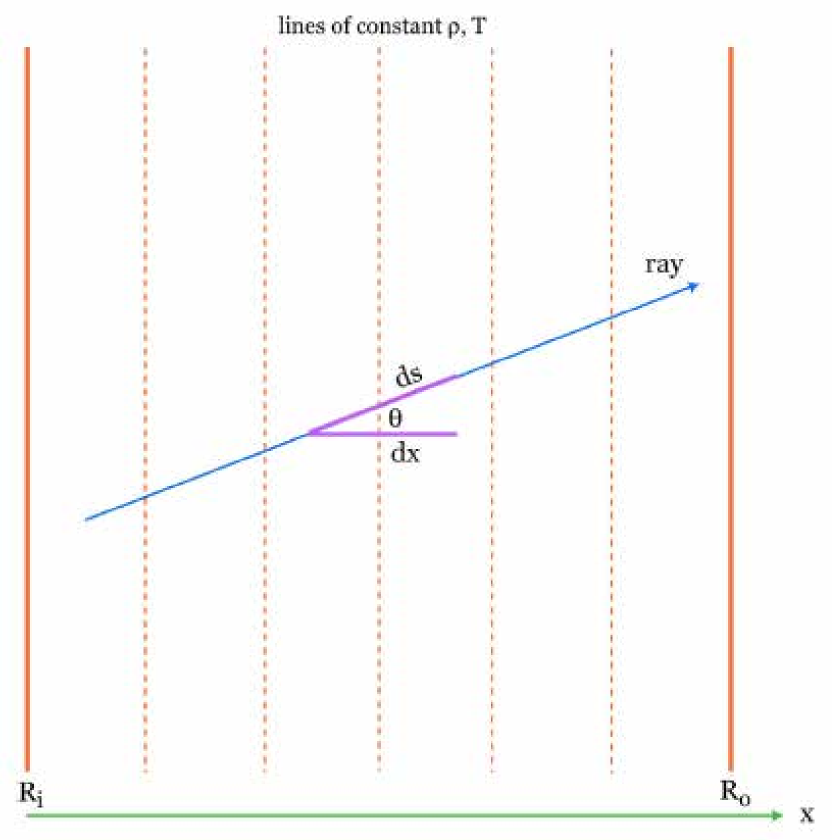
- From (5.3.23) $\mathrm{d}I = -\alpha I \mathrm{d}s$, energy removed by absorption
  - $\mathrm{d}E_\nu^{\mathrm{abs}} = \mathrm{d}I_\nu^{\mathrm{abs}}\mathrm{d}A\mathrm{d}\omega\mathrm{d}\nu\mathrm{d}t = -\alpha_\nu I_\nu \mathrm{d}A\mathrm{d}\omega\mathrm{d}\nu\mathrm{d}t\mathrm{d}s$ (5.3.37)

- The net change in specific intensity
  - $\mathrm{d}I_\nu = -\alpha_\nu I_\nu \mathrm{d}s + \epsilon_\nu \mathrm{d}s$ (5.3.38)
  - $\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + \epsilon_\nu$ (3.5.39)
    - Differential equation describing the flow of radiation through matter



The equation of radiative heat transfer Plane-parallel symmetry

- Plane-parallel symmetry
  - $\mathrm{d}x = \cos\theta \, \mathrm{d}s = \mu \mathrm{d}s$ (5.3.40)
    - $\mu = \cos\theta$ (5.3.41)
  - $\frac{d}{ds} = \mu \frac{\mathrm{d}}{\mathrm{d}x}$ (5.3.42)
- The equation of radiative heat transfer
  - (5.3.39) $\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + \epsilon_\nu$
  - $\mu \frac{\mathrm{d}I_\nu(\mu,x)}{\mathrm{d}x} = -\alpha_\nu(x) I_\nu(\mu,x) + \epsilon_\nu(x)$ (5.3.43)



The equation of radiative heat transfer
Source function & local thermal equilibrium

- In plane-parallel symmetry
  - (5.3.43): $\mu \frac{\mathrm{d}I_\nu(\mu,x)}{\mathrm{d}x} = -\alpha_\nu(x) I_\nu(\mu,x) + \epsilon_\nu(x)$
  - $\frac{\mu}{\alpha_\nu(x)} \frac{\mathrm{d}I_\nu(\mu,x)}{\mathrm{d}x} = -I_\nu(\mu,x) + \frac{\epsilon_\nu(x)}{\alpha_\nu(x)} = S_\nu(x) - I_\nu(\mu,x)$ (5.3.44)
    - Source function: $S_\nu(x) = \frac{\epsilon_\nu(x)}{\alpha_\nu(x)}$ (5.3.45)

- Local thermal equilibrium (LTE)
  - Kirchihoff's law: balance of emission and absorption
    - Emissivity $\varepsilon_\nu(x) = \epsilon_\nu(x)/B_\nu(x)$ (5.3.46)
    - $\xi_\nu(x) \approx \alpha_\nu(x)$ (5.3.47)
  - $S_\nu(x) = \frac{\epsilon_\nu(x)}{\alpha_\nu(x)} = \frac{\varepsilon_\nu(x) B_\nu(x)}{\alpha_\nu(x)} \approx B_\nu(x)$ (5.3.48)

Radiative thermal conductivity -1

- Radiative thermal conductivity under $\mathrm{d}T/\mathrm{d}z$
  - Plane-parallel symmetry: (5.3.43) $\frac{\mu}{\alpha_\nu}\frac{\mathrm{d}I_\nu(\mu,x)}{\mathrm{d}x} = S_\nu(x) - I_\nu(\mu,x)$
  - Local thermal equilibrium: (5.3.46) $S_\nu(x) = B_\nu^{\mathrm{med}}(\nu,T)$, $B_\nu^{\mathrm{med}}(\nu,T)$: $B_\nu(T,x)$ in medium
- Spatial variation of the local intensity:
  - $I_\nu(\mu,x) = B_\nu^{\mathrm{med}}(x) - \frac{\mu}{\alpha_\nu}\frac{\mathrm{d}I_\nu(\mu,x)}{\mathrm{d}x} \approx B_\nu(x) - \frac{\mu}{\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(\nu,T)}{\partial x}$ (5.3.49)
- The energy flux at ν integrated over all direction from (5.3.3) $F_\nu = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_\nu \cos\theta \sin\theta \,\mathrm{d}\theta \mathrm{d}\phi$
  - $F_\nu = \int_{4\pi} I_\nu \mu d\Omega = \int_{4\pi} B_\nu^{\mathrm{med}}(T,x) d\Omega - \int_0^{2\pi}\mathrm{d}\phi \int_0^{\pi} \frac{\mu}{\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(\nu,T)}{\partial x}\cos\theta \sin\theta \,\mathrm{d}\theta$
  - $F_\nu \approx -\frac{2\pi}{\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\int_{-1}^{1}\mu^2 \mathrm{d}\mu = -\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}$ (5.3.50)
    - $\int_{4\pi} B_\nu^{\mathrm{med}}(T,x) d\Omega \approx 0$, $\mu = \cos\theta$ → $\mathrm{d}\mu = -\sin\theta \,\mathrm{d}\theta$

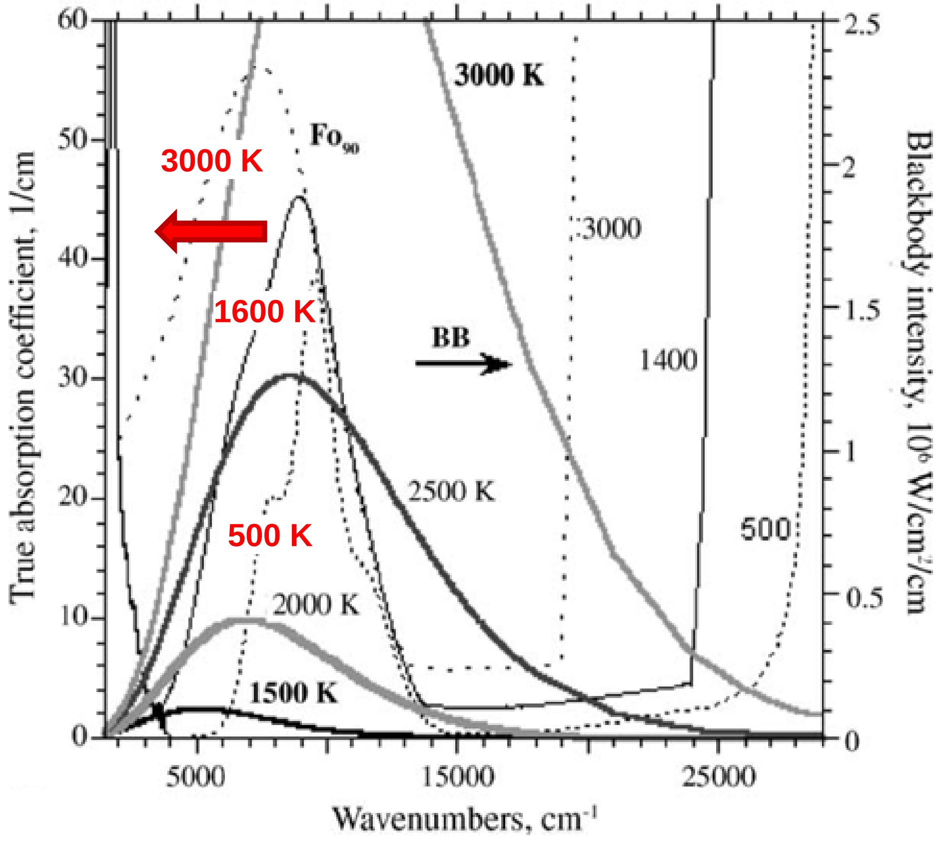
Radiative thermal conductivity -2

- Radiative thermal conductivity under $\mathrm{d}T/\mathrm{d}z$
  - The energy flux at ν
    - (5.3.50) $F_\nu = -\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}$ (5.3.51)
  - Heat flow: integrating the energy flux at ν over all ν
    - $q_{\mathrm{rad}} = \int_0^\infty F_\nu \mathrm{d}\nu = \int_0^\infty \left(-\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu = -\frac{4\pi}{3}\int_0^\infty \left(\frac{1}{\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu$ (5.3.52)
  - Rosseland mean absorption coefficient: $\frac{1}{\alpha_R} = \frac{\int_0^\infty \left(\frac{1}{\alpha_\nu}\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu}{\int_0^\infty \left(\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu}$
    - $q_{\mathrm{rad}} \approx -\frac{4\pi}{3}\frac{1}{\alpha_R}\int_0^\infty \left(\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu$ (5.3.53)

Radiative thermal conductivity -3

- Radiative thermal conductivity under $\mathrm{d}T/\mathrm{d}z$
  - (5.3.52): $q_\mathrm{R} = -\frac{4\pi}{3\alpha_R}\int_0^\infty \left(\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu$
- Radiative thermal conductivity, K_R
  - $k_\mathrm{R} = -\frac{q_\mathrm{R}}{\mathrm{d}T/\mathrm{d}z} = \frac{4\pi}{3\alpha_R}\int_0^\infty \left(\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial x}\right)\mathrm{d}\nu\,\frac{\mathrm{d}z}{\mathrm{d}T} = \frac{4\pi}{3\alpha_R}\int_0^\infty \left(\frac{\partial B_\nu^{\mathrm{med}}(T,x)}{\partial T}\right)\mathrm{d}\nu$ (5.3.54)
- Stefan-Boltzmann law in a medium, (5.3.26) $\int_0^\infty B_\nu^{\mathrm{med}}(\nu,T)\mathrm{d}\nu = \langle n^2\rangle\frac{\sigma}{\pi}T^4$
  - (5.3.54) $k_\mathrm{R} = \frac{4\pi}{3\alpha_R}\frac{\partial}{\partial T}\int_0^\infty B_\nu^{\mathrm{med}}(T,x)\mathrm{d}\nu$
  - $k_\mathrm{R} = \frac{4\pi}{3\alpha_R}\frac{\partial}{\partial T}\left(\langle n^2\rangle\frac{\sigma}{\pi}T^4\right) = \frac{16}{3\alpha_R}\langle n^2\rangle\sigma T^3$ (5.3.55)
    - Clark's (1957) formula $k_R = \frac{16n^2\sigma T^3}{3\alpha_R}$ (5.3.55')

Temperature dependence



Natural olivine, Hofmeister, 2005

- With increasing T, the absorbance α_R increases
- (5.3.55'): $k_R = \frac{16n^2\sigma T^3}{3\alpha_R}$
  - k_R increases less than proportional to T^3
- Frequency-dependent
  - The mean emission shifts to higher wavenumber (higher frequency, shorter wavelength) with T
  - The absorption band shifts to lower wavenumber (lower frequency, longer wavelength) with T

Another thermal resistance

q Grain boundary scattering

Ø Different refractive index à grain-boundary reflection.

ü Different minerals: different refractive index.

ü Different directions of optically anisotropic minerals: different refractive index.

Ø Λ_{GB}: Photon free path due to the grain-boundary scattering

ü $\Lambda_{\mathrm{GB}} = d/R$ (5.3.56)

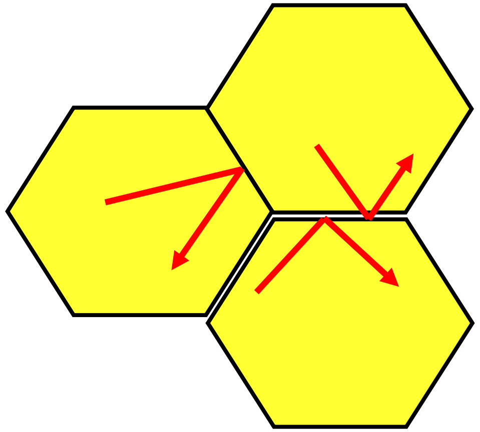
§ d: grain size, R: interface reflectivity (fraction of reflected light)

§ Maybe $R \xrightarrow[\nu\to\infty]{} 0$ because $n \xrightarrow[\nu\to\infty]{} 1$

ØThe total photon mean free path Λ_{tot}

ü $\frac{1}{\Lambda_{\mathrm{tot}}} = \frac{1}{\Lambda_{abs}} + \frac{1}{\Lambda_{\mathrm{GB}}} = \alpha + \frac{R}{d}$ (5.3.57)

§ (5.3.4): $\Lambda_{abs} = 1/\alpha$



Hofmeister's (2005) model -?

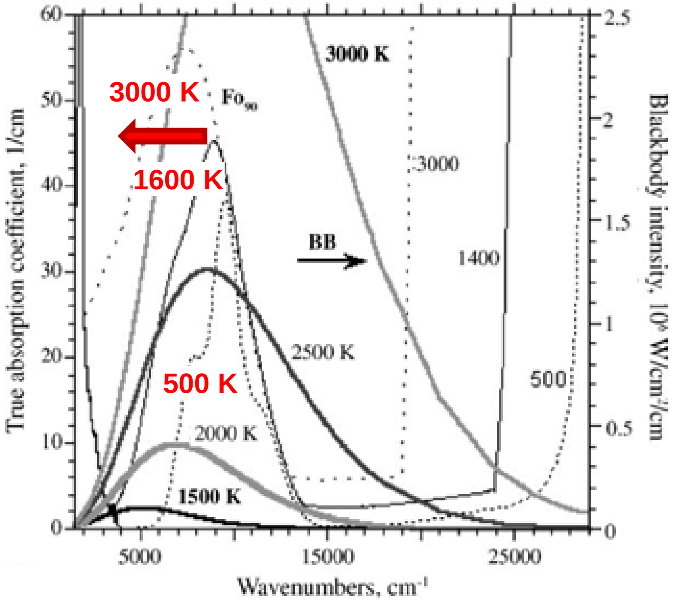
- Kirchhoff's law: the emissivity ε and absorptivity ξ at each wavelength is equal for an arbitrary body emitting and absorbing thermal radiation in thermal equilibrium
  - $\varepsilon = \xi = 1 - \exp(-\alpha s)$ (5.3._)

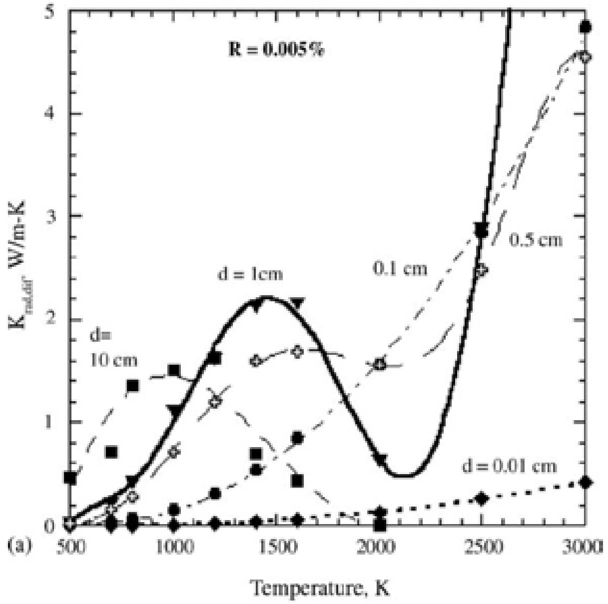
- High absorbance ⇔ high emittance
  - If the surface of material is completely black (i.e. absorbs light completely), it emits thermal radiation by Plank's distribution.
  - If the absorptivity is not 100 %, the emission is lower than the Plank distribution
    - $S(\nu, T) = \xi(\nu) B_\nu(\nu, T) = \{1 - \exp(-\alpha d)\} B_\nu(\nu, T)$ (5.3._)
      - $S(\nu, T)$: the source function, the radiation of a real material
  - The radiative heat transfer of highly transparent material is not necessarily high

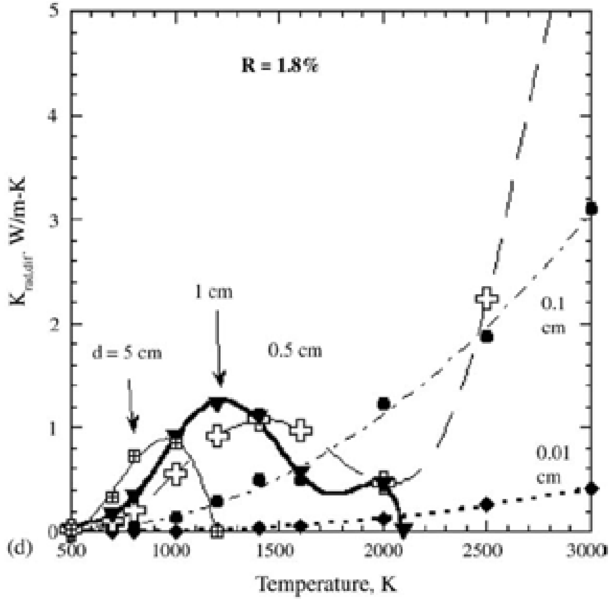
Hofmeister’s (2005) model

- Diffusive radiative thermal conductivity (Hofmeister, 2005)
  - $k_R(T) = \frac{4\pi}{3}\int_0^\infty \frac{d}{R+\alpha(\nu,T)d}\frac{c}{\nu^2}\frac{\partial\{n(\nu,T)^2[1-\exp(-\alpha(\nu,T)d)]B_\nu(\nu,T)\}}{\partial T}\mathrm{d}\nu$ (5.3.__)
- Simplifying (5.3.__)
  - $k_R(T) = \frac{4\pi d n^2}{3}\int_0^\infty \frac{1-\mathrm{e}^{-\alpha d}}{1+\alpha d}\frac{c}{\nu^2}\frac{\partial B_\nu(\nu,T)}{\partial T}\mathrm{d}\nu$ (5.3.__)
  - $k_R(T) \xrightarrow[\alpha\to 0]{} \frac{4\pi d n^2}{3}\int_0^\infty \frac{1-(1-\alpha d)}{1}\frac{c}{\nu^2}\frac{\mathrm{d}B_\nu(\nu,T)}{\mathrm{d}T}\mathrm{d}\nu = \frac{4\pi d^2 n^2}{3}\int_0^\infty \frac{\alpha c}{\nu^2}\frac{\partial B_\nu(\nu,T)}{\partial T}\mathrm{d}\nu \xrightarrow[\alpha\to 0]{} 0$ (5.3.__)
    - No radiative heat transfer in completely transparent materials
      - No energy conversion from photon to the material
  - $k_R(T) \xrightarrow[\alpha\to\infty]{} \frac{4\pi n^2}{3}\int_0^\infty \frac{c}{\alpha\nu^2}\frac{\partial\{B_\nu(\nu,T)\}}{\partial T}\mathrm{d}\nu \xrightarrow[\alpha\to\infty]{} 0$ (5.3.__)
    - No radiative heat transfer in completely opaque materials

Radiative thermal conductivity of polycrystalline olivine (Hofmeister, 2005)







Moderate grain-sizes (mm to cm): local maximum and minimum

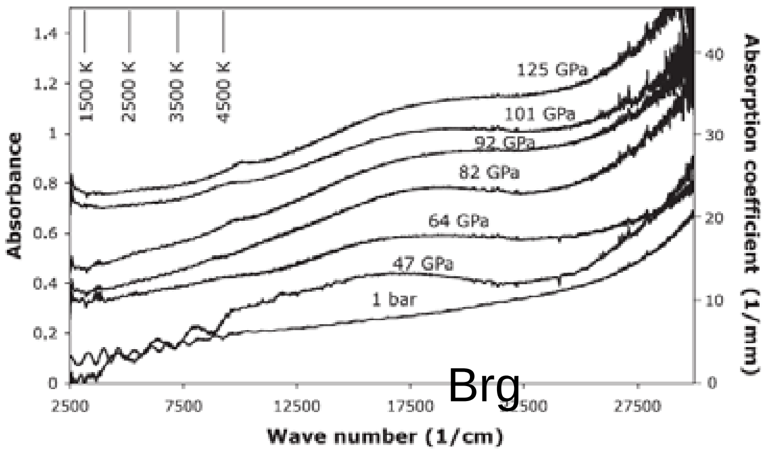
The blackbody peak & near-IR absorption

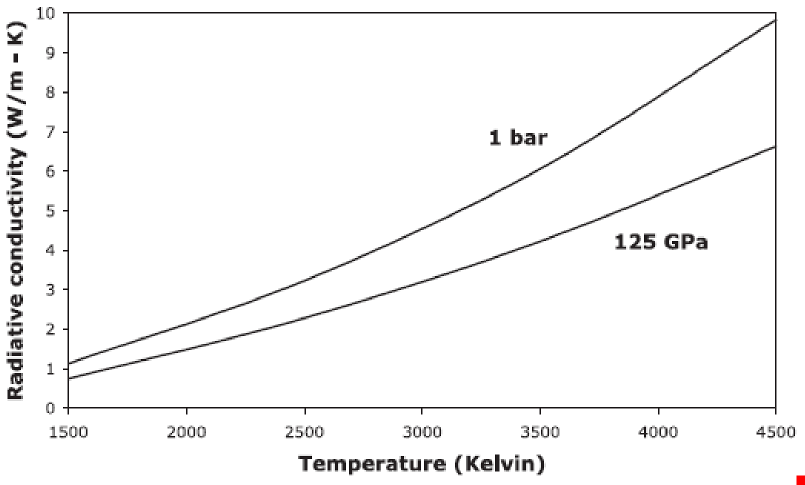
Large grain size (above cm): zero radiative thermal conductivity

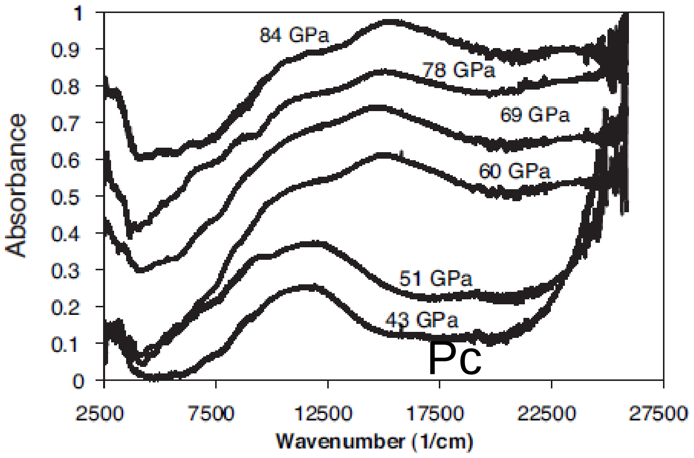
High transmission at high ν at high T

Optical absorption of the lower mantle minerals

- Absorbance increases with increasing pressure
- Brg and Pc have similar absorption coefficient up to 40 /mm
- From these data, the radiative heat transfer in the lower mantle is 3-5 W/mK







Mineral Physics I

Chapter 5. Heat transfer properties

Section 3. Radiative thermal conductivity

End