5.1. Physics of heat transfer

The Earth's interior is essentially hot and characterized by spatially heterogeneous thermal structures. Since lithological materials with higher temperature generally have lower density due to their thermal expansion, the thermal heterogeneity leads to regional buoyancy, and subsequent dynamic motions within the Earth, including slab subduction, plume ascent, and dynamo in the outer core. To consider these problems, it is highly required to understand the heat transfer properties of the Earth's constituents.

1. Classification of heat transfer mechanisms

1.1 Conductive heat transfer

Conductive heat transfer is equivalent to heat flows along the temperature gradient between contacting two bodies. This can be also interpreted as energy transfer by the propagation of atomic vibration because the vibration is more vigorous at higher temperature (T). In this case, the carrier particle of the energy is phonon. The most important equation to describe this phenomenon is Fourier's law, which is, in one-dimensional situations, expressed as below:

$$q = -k\frac{dT}{dx} \tag{5.1.1}$$

where q is the heat flow (transferred heat per unit area and per unit time), k is the thermal conductivity, which is independent of the absolute temperature, and x is the distance.

1.2 Radiative heat transfer

Radiative heat transfer is the energy transfer through emitted light from a high-T body. In this case, the carrier particle of the energy is photon. This process, namely thermal radiation, is described by Stephan-Boltzmann's law (See Section 2 for details).

1.3 Convective heat transfer

Convective heat transfer is different from the two mechanisms above in that it is accompanied by the movement of high-T body. Thermal structure of a given material in completely convective conditions is adiabatic. For thermodynamic properties in such a situation, please refer to the section 1.5: Adiabat.

2. Heat transfer in space

Before considering heat transfer in lithological materials, I first describe radiative heat transfer in 'space' or vacuum (*i.e.* thermal radiation). When a single high-temperature body is located in space, electromagnetic wave is emitted from the body to reach thermal equilibrium. This process follows the Stefan-Boltzmann law as below:

$$\varepsilon = \sigma T^4 \tag{5.1.2}$$

where ε is the total energy of electromagnetic wave emitted from a black body, and σ is the Stefan-Boltzmann constant (5.670367×10^{-8} kg s^{-3} K^{-4}). This equation shows that the total energy of the emitted electromagnetic wave is proportional to the fourth power of temperature.

Then, I discuss heat transfer between parallel surfaces A and B in space. Since energies emitted from the individual surfaces per unit area per time (q_A and q_B) are expressed as $q_A = \sigma T_A^4$ and $q_B = \sigma T_B^4$ when temperatures of them are respectively T_A and T_B, the net heat flow from the surface A to B (q) is:

$$q = q_A - q_B = \sigma T_A^4 - \sigma T_B^4$$

Therefore, if T_A and T_B are similar values (as T_{AB}), q is approximated by a simpler equation as below:

$$\begin{aligned} q &= \sigma(T_A^4 - T_B^4) \\ &= \sigma(T_A^2 + T_B^2)(T_A^2 - T_B^2) \\ &= \sigma(T_A^2 + T_B^2)(T_A + T_B)(T_A - T_B) \\ &\approx \sigma \times 2T_{AB}^2 \times 2T_{AB} \times (T_A - T_B) \propto \sigma T_{AB}^3 \times \Delta T \end{aligned} \tag{5.1.3}$$

where ΔT is the difference of the temperatures of two surfaces ($T_A - T_B$). This equation indicates that the heat flow is proportional to the temperature difference and temperature cubed. Here it should be noted that, in this case, the heat flow is independent from the distance between the individual surfaces. This feature is completely different from that of conductive heat transfer. Specifically, Fourier's law shows the dependency of heat flow on the gradient of temperature, which implies the distance is required to describe the phenomenon. This comparison suggests that there is a specific mechanism to cause such distance-dependency of the heat flow in the case of conductive heat transfer.

3. Thermal diffusion

In this section, I describe fundamentals of thermal diffusion by focusing on a thin plate with thickness δ_x and area S (Figure 1). In this case, the coordinates of left and right sides of the plate can be respectively set as x and $x + \delta_x$. This definition leads to the following expression of heat flows at both sides of the plate: heat flow from the left side into the plate is expressed as $q(x)$; and heat flow from the plate to the right side is expressed as $q(x + \delta_x)$. Therefore, the increase in thermal energy in the plate per unit time (Q) can be expressed as below:

$$\begin{aligned} Q &= S\{q(x) - q(x + \delta_x)\} \\ &= -S\left(\frac{\partial q}{\partial x}\right)_t \delta_x \\ &= -\delta_x S\left(\frac{\partial}{\partial x}\right)_t \left[-k\left(\frac{\partial T}{\partial x}\right)_t\right] \quad \because \text{Fourier's law} \\ &= k\delta_x S\left(\frac{\partial^2 T}{\partial x^2}\right)_t \end{aligned} \tag{5.1.4}$$

From this equation, I derive the thermal diffusion equation. First, increase rate of the temperature at a given position x, or $(\partial T/\partial t)_x$, is expressed as below:

$$\left(\frac{\partial T}{\partial t}\right)_x = \frac{Q}{\rho C \delta_x S}$$

where C is the specific heat per weight. Then, by substituting the equation 5.1.4 for Q,

$$\begin{aligned} \left(\frac{\partial T}{\partial t}\right)_x &= \frac{1}{\rho C \delta_x S} \times k\delta_x S\left(\frac{\partial^2 T}{\partial x^2}\right)_t \\ &= \frac{k}{\rho C}\left(\frac{\partial^2 T}{\partial x^2}\right)_t \end{aligned} \tag{5.1.5}$$

This is the thermal diffusion equation. The coefficient $k/\rho C$ is called thermal diffusivity, and often expressed by using the Greek letter κ.

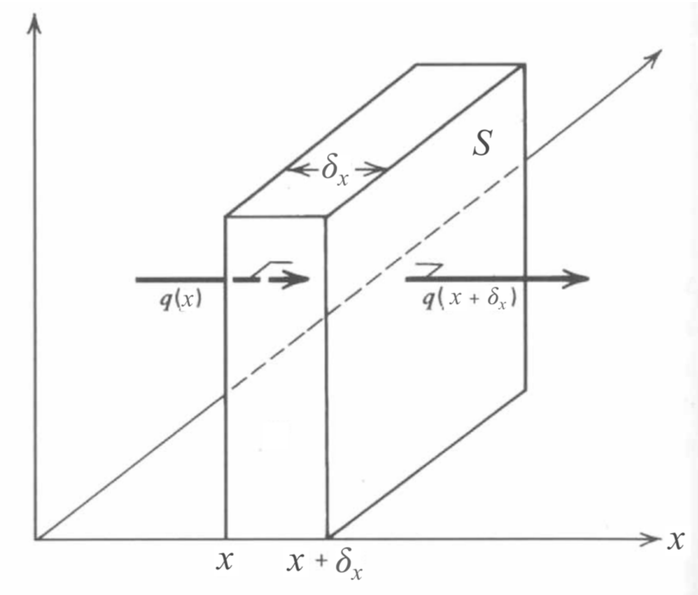

Fig. 1. Schematic diagram of the thin plate.

4. Heat flow and temperature change with time

The equation of thermal diffusion mentioned above denotes the relationship between the temperature heterogeneity, heat flow, and temporal changes of temperature at each point; in other words, once given a certain thermal profile, one can calculate its time evolution. In this section, I mention four examples of spatial thermal distributions and subsequent heat flows, and temperature changing rates.

4.1 Homogeneous temperature

The first example is the case where the temperature is homogeneously high (Figure 2). In this case, because the thermal gradient is zero at every point of the profile, heat flows are calculated to be zero. This implies that there are no temperature changes at any points.

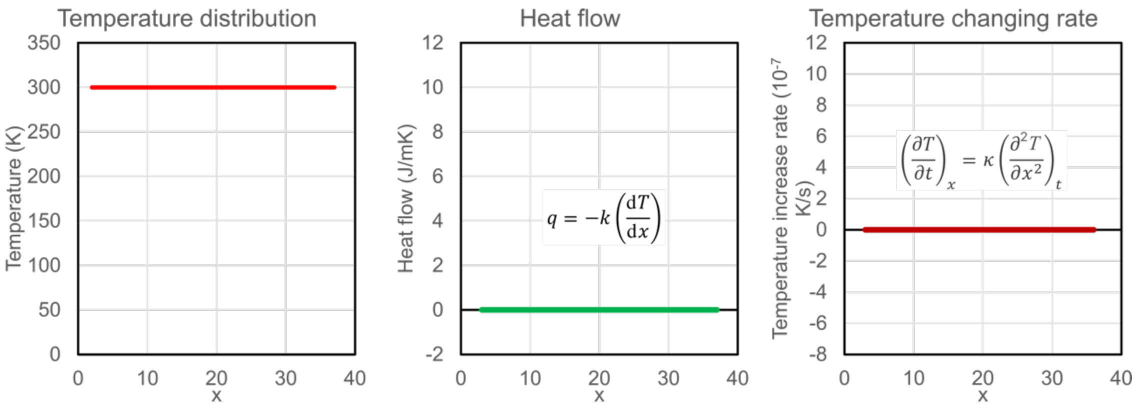

Fig. 2. Relationship among the temperature distribution (left), heat flows (center), and temperature changing rates (right) with a homogeneous thermal profile.

4.2 Linear temperature gradient

The second example is the case where temperature linearly changes according to the distance (Figure 3). In this case, because the thermal gradient is constant at every point of the profile, heat flows are calculated to be constant value (in the example depicted by Figure 3, 25 J $m^{-1}K^{-1}$).

However, because the gradient of heat flows is always zero, there are still no temperature changes at any points.

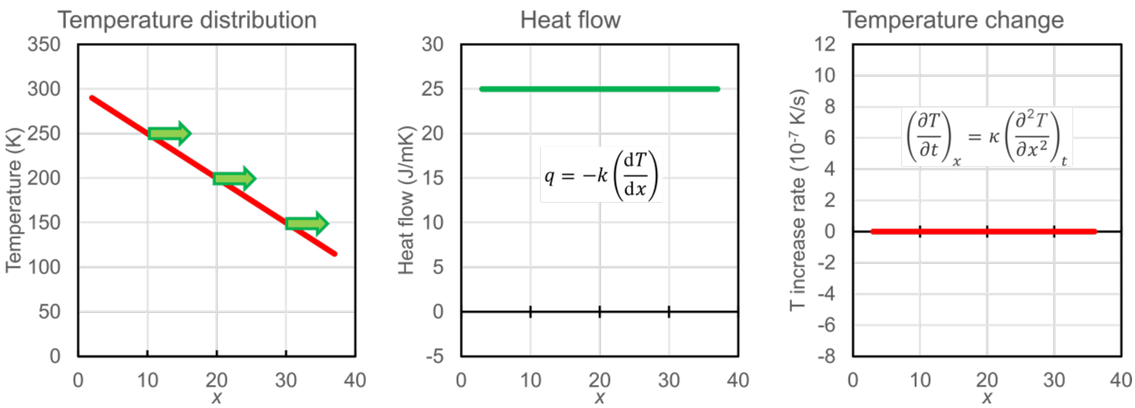

Fig. 3. Relationship among the temperature distribution (left), heat flows (center), and temperature changing rates (right) with a linear thermal profile.

4.3 Parabolic temperature distribution

The third example is the case where the temperature–distance curve is expressed by a parabolic function (Figure 3). In this case, because the thermal gradient is expressed by a linear function of the distance, heat flows are also expressed as a linear function. Notably, heat flow decreases to zero to the bottom (extremum) of the temperature–distance curve. This implies that temperature increases uniformly at every points.

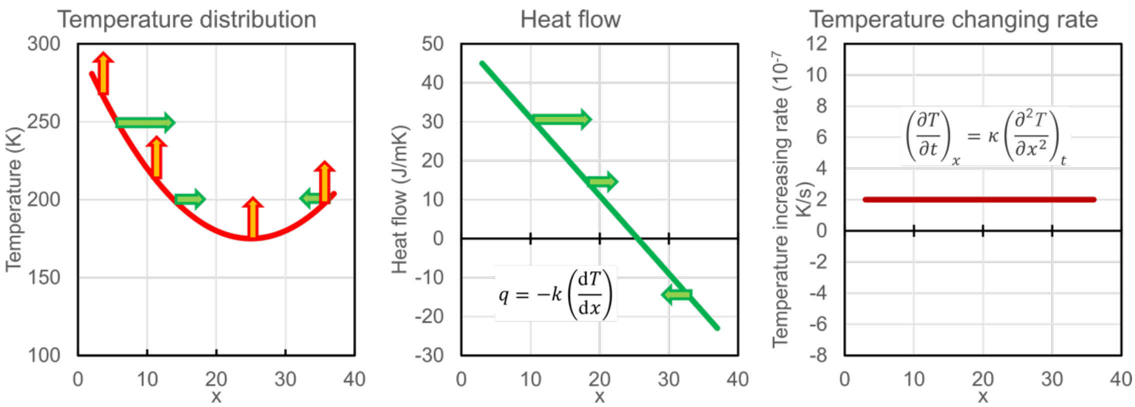

Fig. 4. Relationship among the temperature distribution (left), heat flows (center), and temperature changing rates (right) with a parabolic temperature distribution.

4.4 Varying curvature with position

Finally, I describe more general cases where the curvatures of the temperature–distance curve are diverse (Figure 5). Since the heat flow is a linear function of the thermal gradient, and because the temperature changing rate is a linear function of the second derivative of temperature (*i.e.* a linear function of gradients of the heat flow), the heat flow increases (or decreases) more significantly with increasing (or decreasing) curvature of the temperature distribution. Therefore, temperature increases more rapidly with increase of the curvature of the temperature distribution.

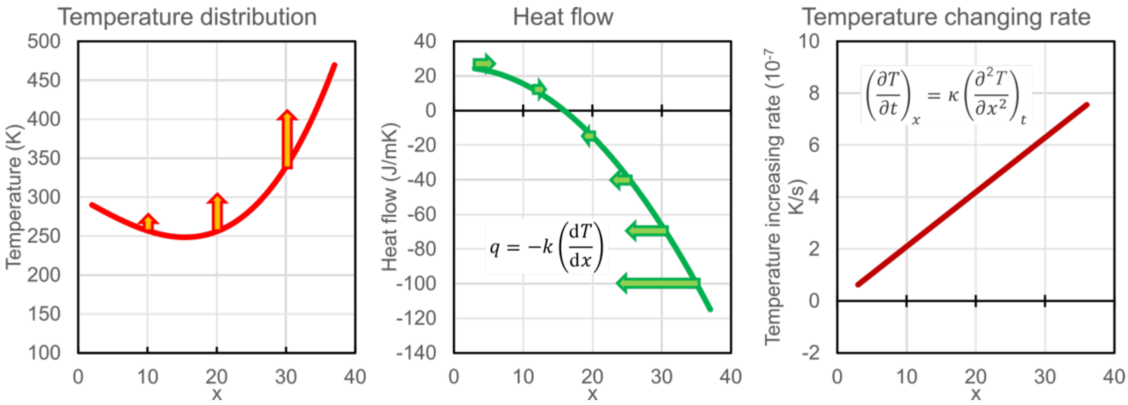

Fig. 5. Relationship among the temperature distribution (left), heat flows (center), and temperature changing rates (right) with a thermal profile whose curvature changes.