Mineral Physics I
Chapter 3. Lattice vibration
Section 5. Phase and group velocities

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Group velocity - 1

q Beat

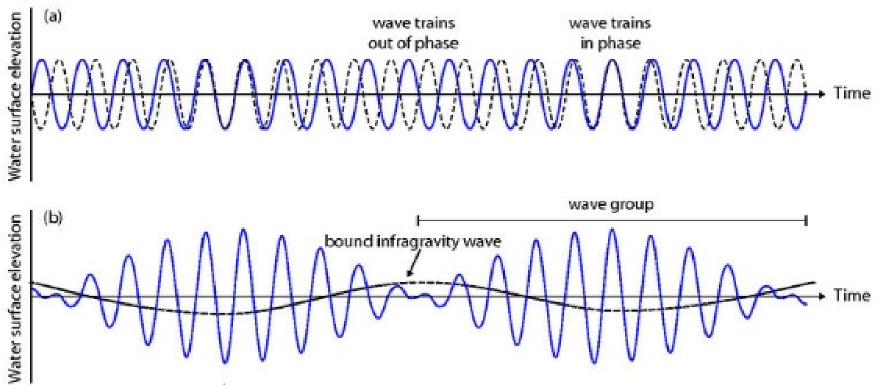
Ø A periodic variation in amplitude as a result of interference between two waves of slightly different frequencies.

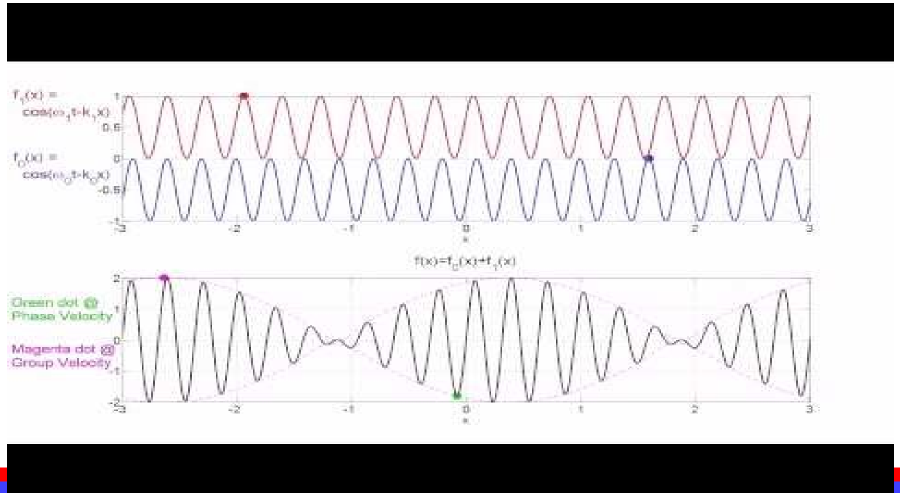
Ø Wave packet

q The velocity of the propagation of the periodic amplitude variation

Ø Group velocity, v_g

q Phase velocity, v_p: the velocity of the phase propagation of each wave





https://www.youtube.com/watch?v=5yiCA5eAcus&t=0s

Group velocity -2

q Two waves (amplitudes are not omitted here):

Ø $f_1(x,t) = \exp[\mathrm{i}(k_1 x - \omega_1 t)], \qquad f_2(x,t) = \exp[\mathrm{i}(k_2 x - \omega_2 t)]$ (3.1.1)

q Superposition of two waves

Ø $f(x,t) = f_1(x,t) + f_2(x,t) = \{\cos[(k_1 x - \omega_1 t)] + \mathrm{i}\sin[(k_1 x - \omega_1 t)]\} + \{\cos[(k_2 x - \omega_2 t)] + \mathrm{i}\sin[(k_2 x - \omega_2 t)]\}$

$$= 2\cos\left[\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right]\cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right]$$

$$+ 2\mathrm{i}\sin\left[\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right]\cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right]$$

$$= 2\cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right]\exp\left[\mathrm{i}\frac{(k_1 x - \omega_1 t) + (k_2 x - \omega_2 t)}{2}\right] \qquad (3.1.2)$$

Ø The factor of the cosine function: beat

Ø The complex exponential function: phase

Group velocity -3

- Beat: amplitude variation, $f_b(x,t)$

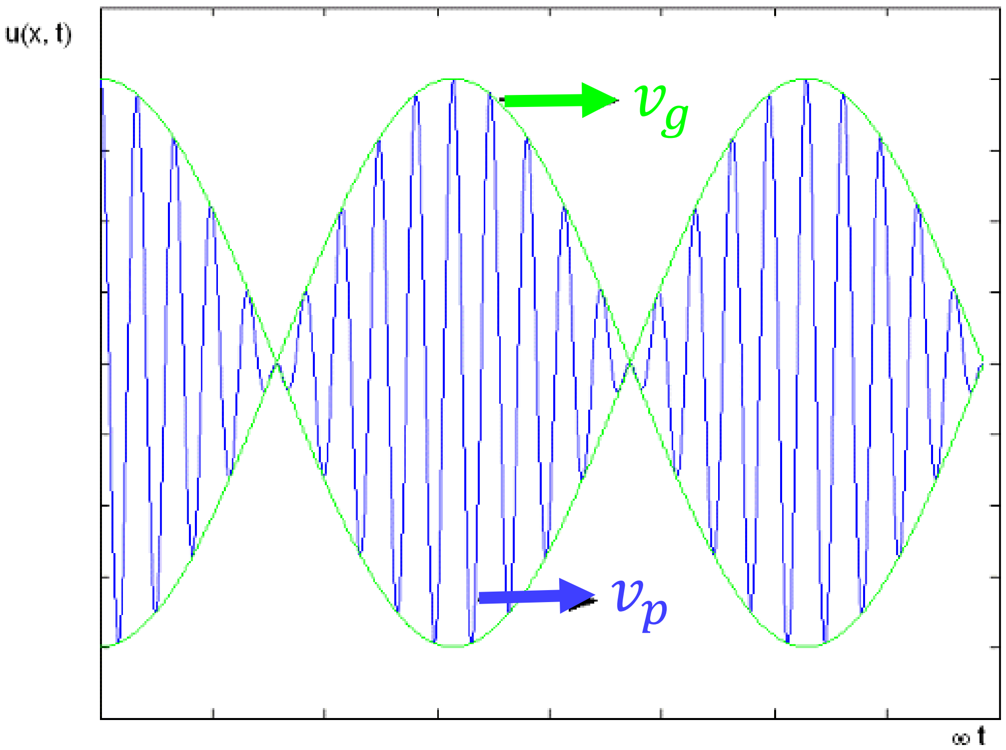
Ø $f_b(x,t) = \cos\left[\frac{(k_1 x - \omega_1 t) - (k_2 x - \omega_2 t)}{2}\right]$

$$= \cos\left[\frac{\Delta k \cdot x - \Delta\omega \cdot t}{2}\right]$$

$$\cong \cos\left[\frac{\Delta k \cdot x - \frac{d\omega}{dk}\Delta k \cdot t}{2}\right]$$

$$\Delta k = k_1 - k_2, \Delta\omega = \omega_1 - \omega_2$$

$$= \cos\left[\frac{\Delta k}{2}\left(x - \frac{d\omega}{dk} t\right)\right] \quad (3.1.3)$$



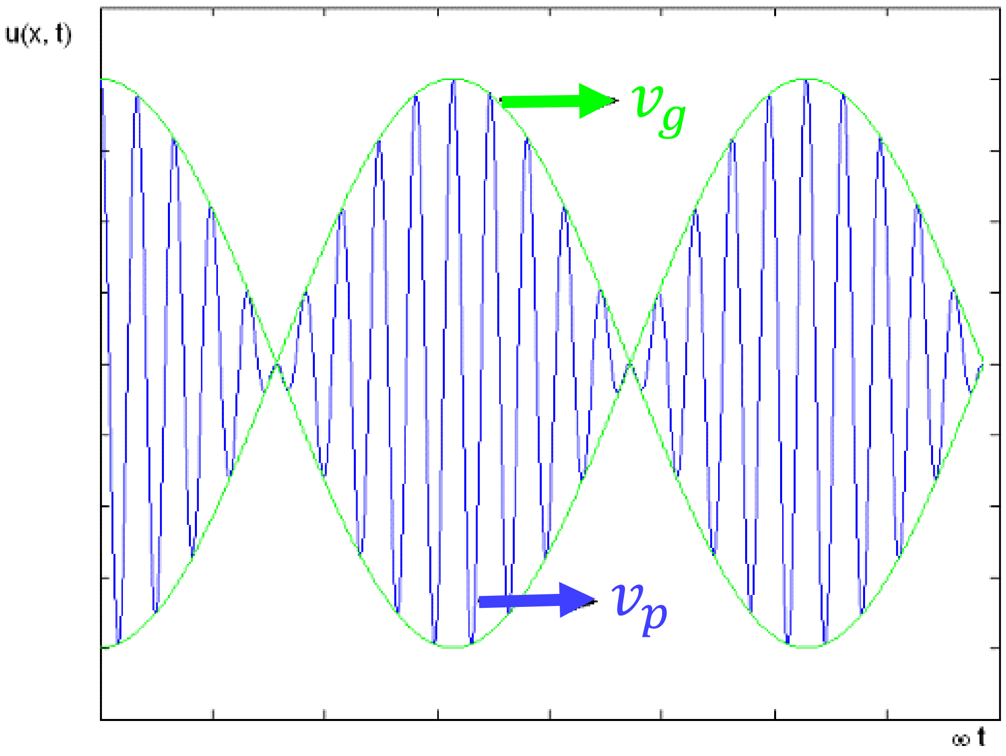
- Group velocity: $v_g = \frac{d\omega}{dk}$ (3.1.4)

Phase velocity

- The oscillation within the beat from Eq. (3.1.2)
  - $f_p(x,t) = \exp\left[i\frac{(k_1x-\omega_1t)+(k_2x-\omega_2t)}{2}\right]$

    $= \exp\left[i\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\right]$

    $= \exp\left[i\frac{k_1+k_2}{2}\left(x - \frac{\omega_1+\omega_2}{k_1+k_2}t\right)\right]$
  - Phase velocity
    - $v_p = \frac{(\omega_1+\omega_2)/2}{(k_1+k_2)/2} = \frac{\bar{\omega}}{\bar{k}}$ (3.1.5)
      - $\bar{\omega}$: average angular frequency
      - $\bar{k}$: average angular wave number



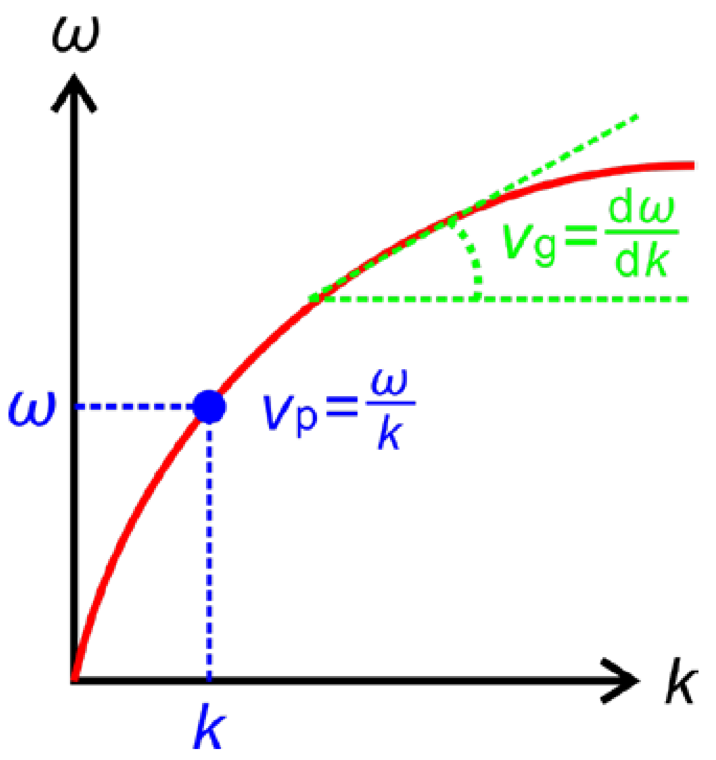
Dispersion relation

q Dispersion relation

Ø $\omega = f(k)$ (3.1.6)

q Group velocity: $v_g = \frac{d\omega}{dk}$ (3.1.4)

q Phase velocity: $v_p = \frac{\omega}{k}$ (3.1.5)



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